NETS AND FILTERS IN TOPOLOGY

by

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ABSTRACT

This paper represents an attempt to present the fundamental theory of nets and filters and to show the important role of the various types of nets and filters in topology. This paper will also attempt to give the relationships between various types of nets and filters. It also deals with O-filters and presents an easy proof of the Tychonoff product theorem. Both a completion and a compactification for a uniformizable space are given in detail.

INTRODUCTION

Why is a net an important concept in mathematics? As a partial answer to this question, we will first try to understand why it was introduced.

As every student of mathematics knows, the concept of a limit is fundamental in analysis. By the time one reaches advanced calculus perhaps one begins to wonder how many disguises the concept of a limit can assume. The saving feature is of course that many of these definitions have strong resemblances. In 1922 Moore and Smith [29] introduced a generalized limit in order to combine many of the important limiting concepts of analysis into a unique theory such that the different types of limiting processes would occur as special cases. Essentially, a net is this generalized limit of Moore and Smith.

Let us now consider the limit of a sequence, the limit of a function and the limit of the sum which gives rise to the Riemann integral and see how they can be grouped under one general definition.

Definition A. A Sequence (a_n) $(n \in J)$ of real numbers converges to a limit a, in notation, $\lim\limits_{n \to \infty} a_n = a$, if and only if for every real number $\epsilon > 0$ there exists an integer n_ϵ such that $|a_N - a| < \epsilon$ for every $N \geq n_\epsilon$.

Definition B. A function $f(x)$ converges to b as x approaches a, in notation, $\lim_{x \to a} f(x) = b$ if and only if for every real number $\epsilon > 0$ there exists a positive number δ_ϵ such that $|b - f(x)| \leq \epsilon$ for every x such that $0 < |x - a| \leq \delta_\epsilon$.

Notation - Let D be any partition of the interval $a \leq x \leq b$ into subintervals $x_{i-1} \leq x \leq x_i$ $(i = 1, \ldots, n)$ such that $a = x_o < x_1 < \ldots < x_n = b$. We call the length of the longest of the subintervals $x_{i-1} \leq x \leq x_i$ the norm of the partition D. Then any bounded function $f(x)$ defined on $a \leq x \leq b$ gives rise to f as a function of D as follows:

$$f(D) = \sum_{i=1}^{n} f(\xi_i)\,(x_i - x_{i-1})$$

where $$x_{i-1} \leq \xi \leq x_i \qquad (i = 1, \ldots, n) .$$

Definition C. $f(D)$ converges to the limit I as the norm of D approaches zero if and only if to every $\epsilon > 0$ there corresponds a positive δ_ϵ such that $|I - f(D)| < \epsilon$ for every D of norm $\leq \delta_\epsilon$.

The limit I if it exists, is, of course, the Riemann integral of $f(x)$ on the interval $a \leq x \leq b$.

These three definitions seem at first sight to be quite different. However, definitions B and C can be restated, as shown by Smith [41], to look more like definition A.

Definition B'. Let X be the set of all points different from a. Then $\lim f(x) = b$ if and only if for every ϵ there is an x_ϵ in X such that $|b - f(x)| < \epsilon$ for every $x \in X$ such that $|x - a| \leq |x_\epsilon - a|$.

Definition C'. If D, $f(x)$ and $f(D)$ are as previously indicated and if we say that a partition D_1 of $a \leq x \leq b$ is finer than a partition D_2 of that interval whenever everyone of the intervals of which D_1 is made up is a subinterval of some interval of D_2, then $f(D)$ converges to I provided for every $\epsilon > 0$ there is a partition D_ϵ such that $|I - f(D)| \leq \epsilon$ for every partition D finer than D_ϵ.

It follows that definitions B and B' and definitions C and C' are equivalent.

In order to combine the three definitions under a more general definition, we now look closer at the essential elements of the definitions of the limit of a sequence. To begin with the property that n is an integer is quite irrelevant since the same kind of definition applies as x approaches infinity over the whole real number system. But the relationship that n is after n_ϵ is important. Two other properties of the integers are also important: these are, transitivity and given any two integers a and b there exists a third c, which is larger than both a and b. In definitions A, B', C' we have a real valued function defined for every element of a certain class and in each case the function converges to a real valued function provided

Definition F. [illegible]

Definition G. [illegible]

We wish to compare the above definitions. [illegible]

certain conditions are met. This suggested to Moore and Smith an abstract theory of which these definitions are instances.

Let S be a collection of elements and let R be a relationship between certain pairs of S. That is, we assume that if S_1, S_2 are a pair of elements of S, then we know whether S_1 is, or is not, R related to S_2. If S_1 is R related to S_2 we write $S_1 R S_2$. We will also require that R be a transitive relationship on S and if p_1 and p_2 are elements of S then there exists an element $p_3 \in S$ such that $p_3 R p_1$ and $p_3 R p_2$. With these requirements on S Moore and Smith [29] gave the following:

Definition D. If $f(s)$ is a real valued function defined for every $s \in S$, then $f(s)$ converges to a, in notation, $\lim f(s) = a$, if and only if for every positive ϵ there is an element $s_\epsilon \in S$ such that $|a - f(s)| \leq \epsilon$ for every $s \in S$ such that $s R s_\epsilon$.

It is easily seen that definitions A, B' and C' are special cases of definition D. A net and its limit is essentially, as we shall see later, this generalized limit of Moore and Smith, except f is not restricted to being a real valued function.

A second reason why nets are important in mathematics is because sequences are inadequate in problems involving topological convergence. For example, consider the set X of ordinals less than or equal to the first uncountable ordinal, Ω, with the ordered topology. Ω is

a cluster point of this set but there is no sequence in X which converges to Ω. But it is possible to extract a net which converges to Ω.

At first glance, the reasons for introducing filters in topology seems to be quite different from those for nets. Filters grew mainly from the finite intersection property and the neighbourhood axioms. Topology deals essentially with infinite sets, while it is much easier to operate with finite sets. It is therefore necessary to have a tool which passes from the finite to the infinite and conversely. The necessary tool has to have finite features in its definition, but to be infinite in essence; and filters satisfy both requirements.

H. Cartan is considered as the 'father' of filters - although they seem to have been discovered simultaneously and independently by Birkoff [10], Cartan [17] and Smith [41]. It was Cartan who attached the name and developed most of the fundamental theory. It was shown by Bartle that as far as convergence properties are concerned nets and filters are equivalent notions.

CHAPTER I

NETS AND FILTERS

A. NETS

1.1 Let A be a collection of elements. If we have a relationship $\geq$ defined on the members of A such that

i) $a \geq a$ for all $a \in A$;

ii) If $a_1 \geq a_2$ and $a_2 \geq a_3$, then $a_1 \geq a_3$;

iii) for any pair a_1 , a_2 in A , there is an a_3 in A such that $a_3 \geq a_1$ and $a_3 \geq a_2$,

we then say $\geq$ directs A . A is called a directed set, which is denoted by $(A , \geq)$.

It is primarily condition (iii) which is due to Moore - Smith [29] and which distinguishes directed sets from partially ordered sets. This condition they referred to as the composition property.

1.2 Definition. A net (x_a) $(a \in A)$, in an abstract space X , is a function on the directed set $(A , \geq)$ with values in X .

This definition is in agreement with Kelley [26] and Bartle [4]. It should be noted that some authors choose to refer to what we have called a 'net' as a 'directed set' or a 'directed system'. The term 'net' was first used by Kelley [25] and was supposedly suggested by Norman Steenrod.

Remark: It is immediately obvious that a net (x_a) $(a \in A)$ is a sequence, in the usual sense, if and only if A is the set of positive integers.

B. CONVERGENCE OF NETS

In 1921 Moore and Smith [29] discussed the properties of this generalized sequence and gave nine necessary and sufficient conditions for the existance of a limit. But their work was confined to the case when the function referred to in definition 1.2 was real valued. Birkoff [10] in a paper published in 1937 was the first to consider the convergence of a 'directed set' in a general topological space. Later Tukey [43] introduced the concepts of cofinal and residual directed systems and proved some related properties - but he was mainly interested in the convergence of stacks. Kelly [25] later introduced the concepts of subnets and universal nets.

Birkoff [10] introduced the following definition:

1.3 Definition. A net (x_a) $(a \in A)$ is eventually in a set $B \subset X$ if and only if there is an element a^* of A (depending on B) such that if $a \geq a^*$, then $x_a \in B$.

1.4 Definition. A net $(x_a)(x_a \in A)$ is frequently in a set $B \subset X$ if and only if for each $a^* \in A$ there is an $a \in A$ such that $a \geq a^*$ and $x_a \in B$.

1.5 Definition. A point x in a topological space (X,T) is a cluster point of a net (x_a) $(a \in A)$ if (x_a) $(a \in A)$ is frequently in every neighbourhood of x.

1.6 Definition. A net (x_a) $(a \in A)$ in a topological space (X,T) is said to converge to a point $x \in X$ if and only if the net is eventually in every neighbourhood of X.

Birkoff [10] showed that closure and neighbourhood system can be characterized in terms of nets. In this direction Birkoff established the relationship between convergence and openness (and hence closure) with the following:

1.7 Theorem. A subset S of a topological space (X,T) is open if and only if no net of points outside of S converge to a point of S.

Proof. Suppose S is open. Then S is a neighbourhood of each of its points and any net which converges to $x \in S$ must be eventually in S.

Conversely, suppose S is not open. Then the union of all open subsets of S must exclude at least one $x \in S$. Let U_a be any open set containing this x. Then there must exist $x_a \in U_a$ such that $x_a \notin S$. Order the indices 'a' as follows: $a \geq b$ means $U_a \subset U_b$. Then these indices will form a directed set, $(A, \geq)$. Clearly we can extract a net (x_a) $(a \in A)$ which converges to X. Hence we have found a net $(x_a)(x_a \in A)$ outside of S which converges to x.

An equivalent theorem has been stated by Kelley [26].

1.8 Theorem. Let (X,T) be a topological space. Then:

(a) A point x is a cluster point of a subset of X if and only if there is a net (x_a) $(a \in A)$ in $B - \{x\}$ which converges to x.

(b) A point x belongs to the closure of a subset B of X if and only if there is a net (x_a) $(a \in A)$ in B which converges to x.

(c) A subset B of X is closed if and only if no net in B converges to a point of $X - B$.

C. SUBNETS

Tukey [43] considered two types of subsets of a directed set - cofinal and residual.

1.9 Definition. Let $(A, \geq)$ be a directed set. A subset B of A is called cofinal in A if for each $a \in A$ there exists $b \in B$ such that $b \geq a$.

1.10 Definition. If $(A, \geq)$ is a directed set a subset B of A is residual in A if $\exists\, a' \in A$, such that for all $a \geq a'$ $a \in B$.

Tukey also proved the following propositions which are easily established.

1. B is cofinal in A if and only if $A - B$ is not residual in A. Hence B and $A - B$ cannot both fail to be cofinal in A.

2. Every residual subset B of A is cofinal in A.

3. If A is a directed set and B is cofinal in A, then B is a directed set.

4. If C is cofinal in B and B is cofinal in A, then C is cofinal in A.

5. If $B' \supset B''$, and B'' is cofinal in A, then B' is cofinal in A.

Cofinal directed sets and Residual directed sets immediately suggest similar definition for subnets.

1.11 Definition. Let (x_a) $(a \in A)$ and (y_b) $(b \in B)$ be nets on a set X. Suppose d is a function which maps B into A such that

(i) $y_b = x_{d(b)}$ for all $b \in B$.

(ii) given any $a_0 \in A$, there is a $b_0 \in B$ such that if $b \geq b_0$ then $d(b) \geq a_0$;

then (y_b) $(b \in B)$ is a (cofinal) subnet of (x_a) $(a \in A)$.

Defn. 1.10 is due to Kelly [25] and is widely used.

1.12 Definition. A net (y_b) $(b \in B)$ is a residual subnet of a net (x_a) $(a \in A)$ in a space X if B is a Residual directed subset of A.

It is to be understood that the domains A and B in definition 1.11 may be totally unrelated, but the range (x_a) $(a \in A)$ contains the

(y_b) $(b \in B)$ and as b grows large so does $d(b)$.

Remark: 1. It follows immediately from the definition that if a net is eventually in a given set so is every (cofinal) (residual) subnet.

2. If a net converges to a point so does every (cofinal) (residual) subnet.

Another definition of a subnet has been introduced by WARD [46] . His new definition is as follows:

1.13 <u>Definition.</u> A net (x_a) $(a \in A)$ is a subnet of a net (y_b) $(b \in B)$ if and only if for each n in A , there exists an m in B with the property that if $p \geq m$, then there exists some $N \in A$ such that $N \geq n$ and $x_N = y_p$.

1.14 <u>Lemma.</u> Let (x_a) $(a \in A)$ be a net and let (y_b) $(b \in B)$ be a subnet of (x_a) $(a \in A)$ in the sense of 1.13. Then (y_b) $(b \in B)$ is a subnet of (x_a) $(a \in A)$ in the sense of definition 1.11.

Proof: obvious

The converse to lemma 1.14 is false. In fact the following example shows that the converse need not hold even in the special case when (x_a) $(a \in A)$ and (y_b) $(b \in B)$ are both sequences. Ward [26] asserts that the converse to the lemma does hold when (x_a) $(a \in A)$ and (y_b) $(b \in B)$ are sequences.

Let $X = (x_i)(i \in J^+)$ and $Y = (y_i)(i \in J^+)$ be sequences of distinct terms in a space S. Let $x_i = y_{i+1}$ for all $i \in J^+$ and $y_1 \neq x_i$ for all i. Then clearly Y is a subnet (sequence) of X in the Ward sense. But Y is not a subnet of X in sense of definition 1.11. Ward uses this new definition of a subnet to strengthen the closure results of Kelly [25].

1.15 Definition. [Kelly] A net (x_a) $(a \in A)$ is universal in X if for every subset A of X, then (x_a) $(a \in A)$ is either eventually in A or $X - A$.

It is clear that a universal net converges to each of its cluster points (if they exist).

1.16 Lemma. If f is a function on X to Y and (x_a) $(a \in A)$ is a universal net in X then $f(\,(x_a)\,)$ is universal in Y.

Proof: Let $B \subset Y$. Then $f^{-1}(B)$ is a subset of X and $(x_a)(a \in A)$ is eventually in $f^{-1}(B)$ or $X - f^{-1}(B)$. Therefore $f(\,(x_a)\,)$ is eventually in B or $Y - B$.

Birkoff [10] has given the following relationship between nets and continuous functions:

1.17 Theorem. Let X, Y be topological spaces. A function f from X to Y is continuous if and only if for every net (x_a) $(a \in A)$ converging to $x \in X$, the net $(f(x_a))$ $(a \in A)$ converges to $f(x)$ in Y.

Proof: The necessity of the condition is obvious.

Conversely, if $(f(x_a))$ $(a \in A)$ does not converge to $f(x)$ for some net (x_a) $(a \in A)$ convergent to x, then there exists an open set $U \subset Y$ such that U is a neighbourhood of $f(x)$ and $(f(x_a))$ $(a \in A)$ is frequently outside U. Therefore there exists no open set $V \subset X$ such that V is a neighbourhood of x and $f(V) \subset U$. Hence f must be continuous.

We shall now turn to a discussion of filters. The presentation shall follow Bourbaki and Gaal. Much of the work done on filters and nets will be developed simultaneously.

D. FILTERS

1.18 Definition. (Cartan [17]) A filter $\mathcal{F}$ is a non-empty family of subsets of a space X satisfying the following conditions:

(i) $\emptyset \notin \mathcal{F}$,

(ii) if F_1 and $F_2 \in \mathcal{F}$ then there exists $F_3 \in \mathcal{F}$ such that $F_3 \subseteq F_1 \cap F_2$,

(iii) if $F_1 \in \mathcal{F}$ and $F \supset F_1$ then $F \in \mathcal{F}$.

Other 'types' of filters have been defined similarly. For example Kowalsky [27] defines a filter as above except he does not stipulate that the empty set is not a member.

Thampuran [42] gives a generalization of the filter introduced by Cartan and deals with some properties of this generalized filter. Thampuran gives the following:

1.19 <u>Definition</u>. Let X denote a space and $\emptyset$ the empty set. A family $\mathcal{F}$ of subsets of X is an extended filter in X if and only if $A \in \mathcal{F}$ and $A \subset B$ implies $B \in \mathcal{F}$.

$\mathcal{F}$ is a proper extended filter if and only if $\emptyset \notin \mathcal{F}$ and is an improper extended filter if and only if $\emptyset \in \mathcal{F}$. He does not require that the intersection of two members of $\mathcal{F}$ be a member of $\mathcal{F}$, claiming "most of the results do not depend on this condition".

For most of the work involving filters we actually do not need the 'whole' filter but rather a 'base' for the filter is sufficient.

1.21 <u>Definition</u>. Let $\mathcal{F}$ be a filter on X. A non-empty family $\mathcal{B}$ of $\mathcal{F}$ is a <u>filter base</u> for $\mathcal{F}$ if and only if for each $F \in \mathcal{F}$ there exists $B \in \mathcal{B}$ such that $B \subset F$.

1.22 <u>Lemma</u>. If $\mathcal{B}$ is a filter base on X then:

(1) $\emptyset \notin \mathcal{B}$

(2) If $B_1, B_2 \in \mathcal{B}$ this implies there exists $B_3 \in \mathcal{B}$ such that $B_3 \subseteq B_1 \cap B_2$.

Proof: $\Leftrightarrow$ Follows immediately from the definitions of a filter and a filter base.

Conversely, if $\mathcal{B}$ is a collection of subsets of X satisfying conditions (1) and (2), then there is a unique filter $\mathcal{F}$ on X such that $\mathcal{B}$ is a filter base for $\mathcal{F}$. This filter consists of all sets which contain a set of $\mathcal{B}$. $\mathcal{F}$ is said to be <u>generated</u> by $\mathcal{B}$.

1.23 <u>Definition.</u> Two filter bases $\mathcal{B}_1$ and $\mathcal{B}_2$ are said to be equivalent if they generate the same filter.

1.24 <u>Definition.</u> If $\mathcal{B}_1$ and $\mathcal{B}_2$ are filter bases on X, $\mathcal{B}_1$ is said to be a <u>refinment</u> of $\mathcal{B}_2$ if and only if given $B_2 \in \mathcal{B}_2$ then exists a $B_1 \in \mathcal{B}_1$ such that $B_1 \subseteq B_2$.

It is clear that we might have defined two filter bases $\mathcal{B}_1$ and $\mathcal{B}_2$ to be <u>equivalent</u> if and only if each is a refinment of the other.

Remark: If $\mathcal{B}_1$ is a refinment of $\mathcal{B}_2$ we will write $\mathcal{B}_2 \subseteq \mathcal{B}_1$.

Similarly if $\mathcal{F}_1$ and $\mathcal{F}_2$ are filters on X, $\mathcal{F}_1$ is said to be a refinment of $\mathcal{F}_2$ if and only if $\mathcal{F}_2 \subseteq \mathcal{F}_1$. This relationship partially orders the collection of all filters on X. The filter, whose only element is the set X, is the 'smallest' element in this collection. However, there may not be a largest element. For given

any collection $T = \{F_\alpha \mid \alpha \in I\}$ of filters on X, the intersection $\bigcap_{\alpha\in I} \mathcal{F}\alpha$ is a filter on X which is a lower bound for T. But $\bigcup_{\alpha\in I} \mathcal{F}\alpha$ may not be a filter on X. If T is a linearly ordered set then $\bigcup_{\alpha\in I} \mathcal{F}\alpha$ is a filter on X and is an upper bound for T.

1.25 __Lemma.__ Let f be a mapping from a set X to a set Y. Then $\mathcal{F} = \{ F_\alpha \mid \alpha \in I \}$ is a filter base on X if and only if $\{ f(F_\alpha) \mid \alpha \in I \}$ is a filter base on Y.

Proof: Clear from set theoretical results.

The notion of a filter subbase is sometimes valuable.

1.26 __Definition.__ Let ψ be a collection of subsets of a set X with the finite intersection property. Then the family consisting of all finite intersections of members of ψ is a filter base.

1.28 __Definition.__ A maximal filter is called an __ultra filter__, that is, $\mathcal{F}$ is an ultrafilter if and only if $\mathcal{F} \subset \mathcal{F}' \Rightarrow \mathcal{F} = \mathcal{F}'$.

1.29 __Theorem.__ Every filter is contained in an ultrafilter.

Proof: Let $\mathcal{F}$ be a filter and $\mathcal{C}$ be the collection of all filters finer than $\mathcal{F}$. $\mathcal{C}$ is non empty since $\mathcal{F} \in \mathcal{C}$. Also if $\{\mathcal{F}_\alpha\}$ is a linear ordered subset of $\mathcal{C}$ then $\bigcup \mathcal{F}_\alpha$ is an upper bound of the chain. By Zoins lemma, $\mathcal{C}$ contains a maximal filter.

any collection $\mathcal{F} = \{F_\alpha \mid \alpha \in I\}$ of closed sets in X, the intersection [illegible]

[illegible]

1.27 Theorem. Let f be a mapping from a set X to a set Y. Then $\mathcal{F} = \{F_\alpha \mid \alpha \in I\}$ [illegible] filter base on X [illegible] $\{f(F_\alpha) \mid \alpha \in I\}$ is a filter base on Y.

Proof: [illegible]

[illegible]

1.28 Definition. Let $\mathcal{S}$ be a collection of subsets of a set X with the finite intersection property. Then the family consisting of all finite intersections of members of $\mathcal{S}$ is a filter base.

1.29 Definition. [illegible]

1.30 Theorem. [illegible]

Proof: Let $\mathcal{F}$ be a filter and Γ [illegible]

The following is a well known characterization of ultrafilter by Cartan.

1.30 Theorem. A filter $\mathcal{F}$ is an ultrafilter if and only if $A \cup B \in \mathcal{F}$ implies $A \in \mathcal{F}$ or $B \in \mathcal{F}$.

Proof: Let $\mathcal{F}$ be an ultrafilter and suppose $A \cup B \in \mathcal{F}$, $A \notin \mathcal{F}$.

Define $\mathcal{F}' = \{M \mid M \cup A \in \mathcal{F}\}$. It is easy to verify that $\mathcal{F}'$ is a filter, $\mathcal{F}' \supset \mathcal{F}$ and $B \in \mathcal{F}'$. Hence, by maximality of $\mathcal{F}$, $\mathcal{F} = \mathcal{F}'$ and $B \in \mathcal{F}$.

Conversely, let the condition hold. This implies for every $A \subset X$, either A or $A^c \in \mathcal{F}$. It is clear that if $\mathcal{F}'$ is a refinment of $\mathcal{F}$, then $\mathcal{F} = \mathcal{F}'$.

1.31 Corollary. If $\mathcal{F}$ is an ultrafilter and $A_1 \cup A_2 \cup . . \cup A_n \in$ then $A_i \in \mathcal{F}$.

Remark: It follows immediately from the definition of an ultra filter that if $A \subset X$ and $\mathcal{F}$ is an ultra filter on X such that $A \cap F \neq \emptyset$ for every $F \in \mathcal{F}$ then $A \in \mathcal{F}$.

1.32 Theorem. Every filter is the intersection of all ultrafilters containing it.

Proof: Let $\mathcal{F}$ be any filter. Let $\mathcal{F}'$ be the intersection of all ultrafilters containing $\mathcal{F}$. Clearly $\mathcal{F}' \supset \mathcal{F}$. Suppose

$F' \in \mathcal{F}'$ and $F' \notin \mathcal{F}$. Then F' does not contain any member of $\mathcal{F}$. Hence F'^c intersects every $F \in \mathcal{F}$. Consequently, $\mathcal{F} \cap F'^c$ generates a filter $\mathcal{F}''$ which is finer than $\mathcal{F}$. If U is an ultrafilter containing $\mathcal{F}''$ then $F'^c \in U$. But $F' \in \mathcal{F}' \subset U$. Contradiction.

E. <u>CONVERGENCE OF FILTERS</u>

Bartle [4] has defined a filter $\mathcal{F}$ to be <u>ultimately</u> in a subset A of X if A contains some $F \in \mathcal{F}$.

1.33 <u>Definition</u>. A filter $\mathcal{F}$ in a topological space (X,T) is said to <u>converge</u> to a point $x_0 \in X$ if and only if it is ultimately in every neighbourhood of x_0 (which is equivalent to saying every neighbourhood of x_0 belongs to $\mathcal{F}$).

x_0 is called a limit point of $\mathcal{F}$ and the set of all such points will be denoted by $\lim \mathcal{F}$.

1.34 <u>Definition</u>. A point $x \in X$ is a <u>cluster point</u> of a filter $\mathcal{F}$ on a topological space (X,T) if $N_x \cap F \neq \emptyset$ for every neighbourhood N_x, of x and every $F \in \mathcal{F}$.

The set of all cluster points of a filter $\mathcal{F}$ will be denoted by $\mathrm{cl}\,\mathcal{F}$. It follows from the definition of cluster point that $\mathrm{cl}\,\mathcal{F} = \cap\{\overline{F} \mid F \in \mathcal{F}\}$. From the definition of a filter base $\mathcal{B}$ for a filter $\mathcal{F}$, it is clear that $\mathrm{cl}\,\mathcal{B} = \cap\{\overline{B} \mid B \in \mathcal{B}\}$.

1.35 <u>Theorem</u>. A point x is a cluster point of a filter $\mathcal{F}$ if and only if there is a filter in X which is finer than $\mathcal{F}$ and converges to x.

Proof: Consider $\{N_x \cap F\}$ where N_x is any nbhd of x and $F \in \mathcal{F}$. Clearly this is a filter base which is a refinment of $\mathcal{F}$ and converges to x. .

If $\mathcal{G}$ is a refinment of $\mathcal{F}$ and $\mathcal{G}$ converges to x then clearly $F \cap N_x \neq \emptyset$ for every $F \in \mathcal{F}$ and every neighbourhood N_X of X.

Bourbaki [12] gives the following result:

1.36 <u>Lemma</u>. If $\mathcal{F}$ converges to x then every finer filter also converges to x. In other words $\mathcal{F} \subset \mathcal{F}'$ implies $\lim \mathcal{F} \subset \lim \mathcal{F}'$.

1.37 <u>Lemma</u>. $\mathcal{F}$ converges to x if and only if every ultrafilter containing $\mathcal{F}$ also converges to x.

1.38 <u>Lemma</u>. For every filter $\mathcal{F}$, $\mathrm{cl}\,\mathcal{F}$ is closed.

1.39 <u>Lemma.</u> $\mathcal{F} \subset \mathcal{F}'$ implies $\mathrm{cl}\,\mathcal{F}' \subset \mathrm{cl}\,\mathcal{F}$.

1.40 <u>Lemma.</u> For every filter $\mathcal{F}$, $\lim \mathcal{F} \subset \mathrm{cl}\,\mathcal{F}$.

1.41 <u>Lemma.</u> If U is an ultrafilter in X then $\lim U = \mathrm{cl}\ U$.

The proofs of these lemmas are easy and are therefore omitted.

F. RELATIONSHIP BETWEEN NETS AND FILTERS

The relationship between Moore-Smith convergence and filters was considered by Bruns, Gunter and Schmidt [16] and also by Bartle [4] .

The following theorem by Bartle [4] shows that given a net on a space X there exists a filter on X such that the filter and the given net have the same limit and cluster points.

1.42 <u>Theorem.</u> Let (X,T) be a topological space and (x_a) $(a \in A)$ be a net on X . Then there is a filter $\mathcal{F}$ on X with the property that $\mathrm{cl}\,\mathcal{F} = \mathrm{cl}\,(x_a)$ $(a \in A)$ and $\lim \mathcal{F} = \lim (x_a)$ $(a \in A)$.

Proof: with the given net in X associate a filter base $\mathcal{B} = \{ B_a \}$ in the following way. For every $a \in A$ let $B_a = \{ x_\alpha \mid \alpha \geq a \}$. The family $\mathcal{B}$ so defined is a filter base.

Let $x \in \lim (x_a)$ $(a \in A)$, then there exists $a \in A$ such that for all $\alpha \in A$ and $\alpha \geq a$ $x_\alpha \in N_x$ for each $N_x \in \mathcal{N}(x)$. Therefore $B_\alpha \leq N_x$ and the filter base converges.

Conversely, if $x \in \lim \mathcal{B}$, then given any nbhd N_x of x there exists some $B_\alpha \in \mathcal{B}$ such that $B_\alpha \leq N_x$. Hence $x_\gamma \in N_x$ for all $\gamma \geq \alpha$ and (x_a) $(a \in A)$ converges to x. Hence $\lim (x_a)$ $(a \in A)$ $=$ $\lim \mathcal{B}$.

Similarly it can be shown that the cluster point of the net and the associated filter base are equivalent.

Also given any filter $\mathcal{F}$ on a topological space (X,T) there exists a net on (X,T) which has the same limit and cluster points.

1.43 Theorem. Let $\mathcal{F}$ be a filter on a topological space (X,T). Then there exists a net (x_a) $(a \in A)$ on X such that $\lim (x_a)$ $(a \in A) = \lim \mathcal{F}$ and $\mathrm{cl}\ (x_a)$ $(a \in A)$ $\mathrm{cl}\ \mathcal{F}$.

Proof: Let A be the set of all pairs $a = (x,F)$, where $x \in F \in \mathcal{F}$. Order A by requiring $(x,F) \geq (x',F')$ if and only if $F \subseteq F'$. Clearly A is a directed set. The mapping $(x,F) \to x_a$ gives a net (x_a) $(a \in A)$ on X.

Let x be a limit point of $\mathcal{F}$. Then given any nbhd N_x of x there exists $F \in \mathcal{F}$ such that $F \subseteq N_x$. Clearly (x_a) $(a \in A)$ converges to x.

Conversely, let $x \in \lim (x_a)$ $(a \in A)$. Then given any nbhd N_x of x there exists $a \in A$ such that if $\beta \geq a$, implies $x_\beta \in N_x$. $\beta = (x,F)$. Then $F \subseteq N_x$ and $x \in \lim \mathcal{F}$.

Similarly, it is easily shown that x is a cluster point of $\mathcal{F}$ if and only if it is a cluster point of the associated net.

It follows from theorems 1.42 and 1.43 that as far as convergence properties are concerned nets and filters are equivalent notions. Yet, both are of utmost importance in general topology and which ever is used in solving a given problem depends to some degree upon the nature of the problem and the taste of the solver.

It is clear from theorem 1.42 that given a net (x_a) $(a \in A)$ it has associated with it a unique filter-base. Also if $X = (x_a)$ $(a \in A)$ is any net and $Y = (y_b)$ $(b \in B)$ is a subnet of X , then the filter base associated with Y is a refinment of the filter base associated with X .

However, given a filter base $\mathcal{F}$, the associated net $\gamma = (x_a)$ $(a \in A)$ constructed in theorem 1.43 is, by no means, unique. Also a filter base which refines $\mathcal{F}$ may not give rise to a net which is a subnet of γ . But the situation is not completely hopeless. Bartle [5] has shown that if we start with a net and consider the filter base $\mathcal{B}$ associated with it, then there exists a subnet of the original net such that the filter base associated with it is a refinment of $\mathcal{B}$. The theorem is as follows:

Conversely, let [illegible] $\{A_\xi\}$ $(\xi \in \Sigma)$. Then given any open U_p in X there exists $\xi \in \Sigma$ such that [illegible]; because [illegible] $= (\xi, \beta)$. Thus [illegible] and [illegible].

Similarly it is easily shown that if [illegible] is a filter base on [illegible] if and only if [illegible] is a [illegible] base of [illegible].

By [illegible] theorems 1.38 and 1.45 [illegible] [illegible] notions. [illegible] which have [illegible] upon the nature of the problem and the taste of the author.

It is clear from theorem 1.48 that given a net $\{x_\xi\}$ $(\xi \in \Sigma)$, [illegible] the filter base [illegible] $\{x_\xi\}$ $(\xi \in \Sigma)$ [illegible] [illegible] been associated with [illegible].

However, given a filter base [illegible] [illegible] [illegible] which is a subnet of [illegible]. But [illegible] [illegible] [illegible] of [illegible]. The theorem is as follows:

1.44 Theorem. Let $N = (x_a)$ $(a \in A)$ be a net in X and let $\mathcal{B}(N)$ be the associated filter base. Let $\mathcal{D}$ be a refinment of $\mathcal{B}(N)$. Then there exists a subnet $n = (y_\gamma)$ $(\gamma \in C)$ of N which is such that its associated filter base $\beta(n)$ is a refinment of $\mathcal{D}$.

Proof: N is a net on X and $\mathcal{B}(N)$ is the associated filter base. Consider the collection C of all pairs of $A \times D$ of the form $\gamma = (\alpha,F)$ where $x_\alpha \in F \in D$. Order C by requiring $\gamma = (\alpha,F) \leq \gamma' = (\alpha',F')$ if and only if $\alpha \leq \alpha'$ and $F \supseteq F'$. It is easily seen that $(C,\geq)$ is a directed set.

For an element $\gamma = (\alpha,F)$ in C, define $y_\gamma = x_\alpha$. Then the map $\pi : C \to A$ defined by $\pi(\alpha,F) = \alpha$ shows that $n = (y_\gamma)(\gamma \in C)$ is a subnet of N. Let F_0 be an element in D and select α_0 in A. The set $E(\alpha_0)^\cdot = \{ x_\alpha \mid \alpha \geq \alpha_0 \}$ contains some set F_1 in D and there is and α_1 in A such that $x_{\alpha_1} \in F_0 \cap F$. Let $\gamma_0 = (\alpha_1,F_0)$. Then if $\gamma_0 \leq \gamma = (\alpha,F)$, it follows that $x_\alpha \in F \subseteq F_0$ so that $F(\gamma_0) = \{y_\gamma \mid \gamma_0 \leq \gamma\} \subset F_0$.

Hence $\beta(n)$ refines D.

We have already seen that every net has a filter base associated with it and conversely when the given net is universal, we would expect the associated filter base to be an ultra filter since the cluster points and the limit points are preserved. Bartle has shown that such is indeed the case.

1.45 Theorem. (a) If $N = (x_a)$ $(a \in A)$ is a universal net in a set X , then the associated filter base $\mathcal{B}(N)$ is an ultra filter base.

(b) If $\mathcal{B}$ is an ultrafilter base in X , then the associated net, N , is a universal net.

Proof: (a) Let A be an arbitrary subset of X . Then (x_a) $(a \in A)$ is eventually in A or in $X - A$. Therefore the set $E(a) = \{ x_{a'} \mid a' \geq a \}$ is either in A or A complement.

(b) Let N be the associated net. Let A be the arbitrary subset of X . Since B is eventually in A or $X - A$ it follows that N is also eventually in A or $X - A$. Hence N is a universal net.

Theorem 1.29 tells us that every filter is contained in an ultrafilter. Kelley [25] has shown that every net has a universal subnet, this result follows directly from 1.44 and 1.45.

G. EQUIVALENCE OF NETS

When are two nets equivalent? Smiley [40] has defined two nets to be equivalent if and only if the filters they generate are equal. Bartle [6] defines two nets to be equivalent if and only if each is a subnet of the other. We shall see later that if two nets are equivalent in the Bartle sense they are also equivalent in the

Smiley sense, but not conversely.

1.46 Definition. Two filter bases A and B are equivalent if each is a refinment of the other.

H. INDEXED FILTER BASES

It was stated prior to theorem 1.44 that, with the usual construction, the net associated with a filter base is not unique. However, in a later paper, Bartle [6] has given "a different construction which has the advantage of associating a unique net from a given filter". This is done via indexed filter bases.

1.47 Definition. An indexed filter base in a space X is a mapping $\mathcal{B}$ of a non-empty directed set A into the set of non-empty subsets of X such that if $b \geq b'$, then $\mathcal{B}(b) \subseteq \mathcal{B}(b')$.

Every filter base $\mathcal{B}$ forms an indexed filter base in a natural way: the elements of the directed set A are the set $B \in \mathcal{B}$ ordered by inclusion and the mapping $\mathcal{B}$ is the identity mapping. Clearly, an indexed filter base is a special kind of net in the collection of subsets of X. We also see, that a filter base is merely the range of an indexed filter base.

It is also easy to associate with each net in X an indexed filter base. For let $\alpha = (x_a)$ $(a \in A)$ be a net on X. For arbitrary $a \in A$ let $E_a = \{ x_a \mid \alpha \in A, \alpha \geq a \}$. Then the mapping $\mathcal{B}$ which assigns a to E_a is an indexed filter base. This mapping we shall refer to as the <u>indexed filter base associated with the net α</u> and will denote it by $B(\alpha)$.

1.48 <u>Lemma.</u> Let α, β be two nets on X. If β is a subnet of α then $B(\alpha) \subseteq B(\beta)$.

Proof: Let $\alpha = (x_a)$ $(a \in A)$ and $\beta = (y_c)$ $(c \in C)$.

Let E_a be any set in $B(\alpha)$. Then $E_a = \{x_t \mid t \in A, t \geq a\}$. There exists $c \in C$ and a mapping π from C to A such that $\pi(c) \geq a$. Hence if $L_c = \{ y_\gamma \mid \gamma \in C, \gamma \geq c \}$, then $L_c \subseteq E_a$ and $B(\alpha) \subseteq B(\beta)$.

1.49 <u>Lemma.</u> Let α be a subnet of β and β be a subnet of α then the associated filter bases are equivalent.

Proof: Follows directly from lemma 1.48.

It follows, therefore, from lemma 1.49 that if two nets are equivalent in the Bartle sense they are also equivalent in the Smiley sense. But since the converse to lemma 1.48 does not hold, the reverse implication is not true.

Now we shall see how Bartle constructed a unique net from a given filter base. Let $\mathcal{B}$, be a mapping from a directed set A to a space X, be an indexed filter base. Let $C = \{(x,a) \mid x \in \mathcal{B}(a),\ a \in A\}$. Then C is a directed set under the ordering $(x,a) \leq (x',a')$ if and only if $a \subseteq a'$. The mapping which sends $(x,a) \to x$ is a net on X and is said to be the net $N(\mathcal{B})$ associated with the indexed filter base $\mathcal{B}$.

1.50 <u>Theorem.</u> Let $\mathcal{B}$ be an indexed filter base on X, then $\mathcal{B}$ is equivalent to $B(\ N(\mathcal{B})\ )$.

1.51 <u>Theorem.</u> Let α be a net in X then α is equivalent to $N(\ B(\alpha)\ )$.

The proofs of theorems 1.50 and 1.51 are not hard but are tedious and are omitted. See Bartle [6].

These two theorems show that associating a net with an indexed filter base and associating an indexed filter base with a net are, up to equivalence, inverses of each other.

CHAPTER II

A VARIETY OF NETS AND FILTERS

A. O-FILTERS

Robertson and Franklin [36] introduced the concept of an O-net. A net on a space X, being a mapping whose range is contained in X, is independent of the topology on X. However, an O-net, which is a generalization of an O-sequence as introduced by Levine [28], is defined in terms of the topological structure on X.

2.1 Definition. A net (x_a) $(a \in A)$ on a topological space (X,T) is called an O-net if and only if whenever (x_a) $(a \in A)$ is frequently in an open set U, it is eventually in U.

This immediately suggest the following definition for an O-filter:

2.2 Definition. Let $\mathfrak{F}$ be a filter on a topological space (X,T). $\mathfrak{F}$ is an O-filter on X if and only if U or $X - U$ is a member of $\mathfrak{F}$ for every $U \in T$.

2.3 Theorem. $\mathfrak{F}$ is an O-filter on a topological space (X,T) if and only if the associated net is an O-net.

Proof: Let $\mathfrak{F}$ be an O-filter. Let (x_a) $(a \in A)$ be the net associated with $\mathfrak{F}$ (see theorem 1.43). Let B be any open subset of X such

that (x_a) $(a \in A)$ is frequently in B. Then, clearly, $B \in \mathcal{F}$ and hence (x_a) $(a \in A)$ is eventually in B.

Conversely, let (x_a) $(a \in A)$ be an O-net and let $\mathcal{F}$ be the associated filter (see theorem 1.42). Let B be an arbitrary open set in X. If (x_a) $(a \in A)$ is frequently in B then it is eventually in B and clearly $F_{a'} = \{x_{a'} \mid a' \geq a\} \subset B$ for some $a \in A$. Hence $B \in \mathcal{F}$. If (x_a) $(a \in A)$ is not frequently in B then it is eventually in B^c and $B^c \in \mathcal{F}$.

2.4 Lemma. (x_a) $(a \in A)$ is an O-net if and only if whenever (x_a) $(a \in A)$ is frequently in some closed subset F of X, it is eventually in F.

Proof: Assume (x_a) $(a \in A)$ is an O-net and that it is frequently in $F \subset X$ where F is closed. Then $X - F$ is open. Since (x_a) $(a \in A)$ is not eventually in $X - F$, it cannot be frequently in $X - F$. Therefore (x_a) $(a \in A)$ is eventually in F.

Conversely, let $G \subseteq X$ be open. Let (x_a) $(a \in A)$ be a net in X such that it is frequently in G. Since (x_a) $(a \in A)$ is not eventually in $X - G$, where $X - G$ is closed, it cannot be frequently in $X - G$. Hence (x_a) $(a \in A)$ is eventually in G and hence an O-net.

2.5 Lemma. An O-filter $\mathcal{F}$ converges to each of its cluster points.

such $\{x_s\}$ ($s \in S$) is frequently in G. Thus, clearly, $x \in \bar{A}$ and hence $\{x_s\}$ ($s \in S$) is eventually in U.

Conversely, let $\{x_s\}$ ($s \in S$) be convergent to x. [illegible] the associated filter [illegible]. Let U be any [illegible] open set in X. If $\{x_s\}$ ($s \in S$) is frequently [illegible] eventually in U and [illegible] $\{x_s\}$ ($s \in S$) [illegible] $A \subseteq X$. [illegible] $\{x_s\}$ ($s \in S$) [illegible] that x is [illegible] in X and U [illegible].

2.4 Lemma. $\{x_s\}$ ($s \in S$) is [illegible] if and only if $\{x_s\}$ ($s \in S$) is frequently in each [illegible] which is [illegible] in X.

Proof. Assume $\{x_s\}$ ($s \in S$) is [illegible], and that it is frequently in $T \subseteq X$ where T is closed. Then $X - T$ is open. Since $\{x_s\}$ ($s \in S$) is not eventually in $X - T$, it cannot be frequently in $X - T$. Therefore $\{x_s\}$ ($s \in S$) is eventually in T.

Conversely, let $\{x_s\}$ [illegible] be any set in X such that it is frequently in T. Then $\{x_s\}$ ($s \in S$) is not eventually in $X - T$, hence $X - T$ is [illegible] frequently in $X - T$. Hence $\{x_s\}$ ($s \in S$) is eventually in T, [illegible].

2.5 Lemma. [illegible] is [illegible] if and only if [illegible].

Proof: Let U be any open set containing x, where x is a cluster point of $\mathcal{F}$. Since $F \cap U \neq \emptyset$ for every $F \in \mathcal{F}$, clearly $U^c \notin \mathcal{F}$. Hence $U \in \mathcal{F}$ and $\mathcal{F}$ converges to x.

2.6 <u>Lemma</u>. If $\mathcal{F}$ is an O-filter on a topological space (X,T) and f is a continuous function on X to Y, then $f(\mathcal{F})$ is an O-filter on Y.

Proof: If U is an arbitrary open subset of the range space $f^{-1}(U)$ or $x - f^{-1}(U)$ is a member of $\mathcal{F}$. Clearly U or U^c is a member of the image of $\mathcal{F}$ under f.

The converse to the above lemma does not hold as a characterization of continuous function. A function f from X to Y may carry every O-filter on X into an O-filter on Y but f may not be continuous. This is easily seen by taking X to be the set of positive integers and defining a topology T on X such that $G_{2n+1} \in T$ if and only if G is the collection of all integers less than or equal to $2n+1$ where $n \geq 0$. Then take Y to be the set of positive integers and define a topology Λ on Y such that $\{1\} \in \Lambda$ and $H_{2n} \in \Lambda$ if H_{2n} is the collection of all integers $\leq 2n$. Let f be the identity mapping from X to Y. Then f maps every O-filter on X into an O-filter on Y but f is not continuous.

2.7 <u>Lemma</u>. If $\mathcal{F}$ is an O-filter then every refinment of $\mathcal{F}$ is an O-filter.

Proof: Obvious

Clearly every ultrafilter is an O-filter, but the converse is not true. For let X be the space of positive integers with the topology of finite complements. Let B_n be the set of all integers greater than or equal to n. Let $\mathcal{B} = \{ B_n \mid n \in J \}$. Clearly $\mathcal{B}$ is a filter base. If A is any open subset of X, A belongs to the filter associated with $\mathcal{B}$. Hence $\mathcal{B}$ is a base for an O-filter. But $\mathcal{B}$ does not generate an ultrafilter since neither the set of even or odd integers belong to the associated filter.

It was pointed out earlier that every ultrafilter is an O-filter. Under what condition is every O-filter an ultra filter?

2.8 <u>Theorem</u>. If every O-filter on a topological space (X,T) is an ultrafilter then (X,T) is T_0.

Proof: Assume (X,T) is not T_0, this implies for some $x, y \in X$ and $x \neq y$, each is a limit point of the other. Under this assumption we will construct an O-filter which is not an ultrafilter.

Let $\{ x,y \} \in \mathcal{F}$. For every open set $G \in T$, if $\{ x,y \} \in G$ then $G \in \mathcal{F}$. Otherwise $G^c \in \mathcal{F}$. Clearly $\mathcal{F}$ is an O-filter. But since neither $\{ x \}$ nor $\{ x \}^c$ belong to $\mathcal{F}$,

$\mathcal{F}$ is not an ultrafilter.

Thus only one of $\{x\}$ and $\{y\}$ can be a limit point of the other.

2.9 Theorem. Let (X,T) be a topological space in which every singleton is open except $\{y\}$ and $\{x\}$. If at most one of these is a limit point of the other then every O-filter is an ultrafilter.

Proof: Without loss of generality we may assume that every open set containing y also contains x, but there exists an open set N_x containing x which does not contain y.

Let $\mathcal{F}$ be an O-filter on X. We will shown that $\mathcal{F}$ is an ultrafilter.

If $\mathcal{F}$ is not an ultrafilter than there exists $A \subset X$ such that neither A nor $A^c \in \mathcal{F}$. This implies if $p \in X$, $\{p\} \notin \mathcal{F}$. Otherwise, since $p \in A$ or A^c, A or A^c would belong to $\mathcal{F}$.

Claim: one of x and y is in A, and the other is in A^c. For if both were in A^c, say, then A would be open, since every point in A is open. Therefore we may assume $x \in A$ and $y \in A^c$.

Let N_x and N_y be open nbhds of x and y respectively. Pick N_x such that $y \notin N_x$. Clearly $x \in N_y$. Also $N_x \cap A^c \neq \emptyset$. For otherwise, A is open which implies A or $A^c \in \mathfrak{F}$.

<u>Claim</u>: both N_x and N_y belong to $\mathfrak{F}$.

Assume $N_y{}^c \in \mathfrak{F}$. Since $(A \cup N_y)^c \subset A^c$ and is open in A^c and $A^c \notin \mathfrak{F}$, clearly $(A \cup N_y) \in \mathfrak{F}$. But then $(A \cup N_y) \cap N_y{}^c$, which is a subset of A, is also in $\mathfrak{F}$. Hence $A \in \mathfrak{F}$. A contradiction.

Similarly $N_x \in \mathfrak{F}$.

Hence $P = N_x \cap N_y \in \mathfrak{F}$.

If P were a subset of A or A^c then A or A^c would belong to $\mathfrak{F}$.

Let $R = \{ b \mid b \in P \text{ and } b \in A^c \} \subset A^c$

$S = \{ a \mid a \in A, a \in P \text{ and } a \neq x \} \subset A$

Clearly R is open and S is open and $R^c, S^c \in \mathfrak{F}$. This implies $\{x\} = P \cap R^c \cap S^c \in \mathfrak{F}$.

This implies $A \in \mathfrak{F}$ and $\mathfrak{F}$ is an ultrafilter.

2.10 <u>Corollary</u>: Let (X,T) be a topological space in which every singleton is open except $\{x\}$ and $\{y\}$. Every O-filter is an ultrafilter if and only if at most one of these points is a limit point of the other.

The following is immediate from the definition of an O-filter: A filter $\mathfrak{F}$ on a topological space X is an O-filter if and only if for each cover $\{H_i\}$ $1 \leq i \leq n$ of X, where each H_i is either open or closed, $H_i \in \mathfrak{F}$ for some i.

B. CAUCHY FILTERS

2.11 <u>Definition</u>. Let X be a non empty set and $\mathcal{U}$ be a family of subsets of $X \times X$. $\mathcal{U}$ is said to be a <u>Quasi-uniformity</u> on X if and only if

(i) for each $U \in \mathcal{U}$, $\Delta \leq U$,

(ii) if $U \in \mathcal{U}$, then $V \circ V \subset U$ for some $V \in \mathcal{U}$.

(iii) if U and V are members of U, then $U \cap V \in \mathcal{U}$.

(iv) if $U \in \mathcal{U}$ and $U \subset V \subset X \times X$, then $V \in \mathcal{U}$.

The pair $(X,\mathcal{U})$ is then called quasi-uniform space. This definition is due to Csaszar.

2.12 <u>Definition.</u> $\mathcal{U}$ is called a uniform structure for X if satisfies (i) - (iv) of definition 2.11 and

(v) if $U \in \mathcal{U}$ then $U^{-1} \in \mathcal{U}$.

The pair $(X,\mathcal{U})$ is then called a uniform space. This definition is due to A. Weil.

Cauchy filters were first defined by Weil [47].

2.13 <u>Definition.</u> A filter $\mathcal{F}$ on a uniform space $(X,\mathcal{U})$ is said to be <u>cauchy</u> if for every $U \in \mathcal{U}$ there exists $F \in \mathcal{F}$ such that $F \times F \subseteq U$.

Sieber and Pervin [39] have introduced a 'different' definition for a cauchy filter on a quasi-uniform space:

2.14 <u>Definition.</u> Let $(X,\mathcal{U})$ be a quasi-uniform space and let be a filter on X . $\mathcal{F}$ is said to be a cauchy filter if and only if for every $U \in \mathcal{U}$ there exists an $x = x(U)$ such that $U[x] \in \mathcal{F}$.

These two definitions are equivalent on a uniform space.

2.15 <u>Theorem</u>. A filter $\mathcal{F}$ on a uniform space $(X,\mathcal{U})$ satisfies definition 2.14 if and only if it satisfies definition 2.13.

Proof: Let $\mathcal{F}$ be a cauchy filter on $(X,\mathcal{U})$ according to definition 2.14. Let $U \in \mathcal{U}$. Then there exists a symmetric $V \in \mathcal{U}$ such that $V \circ V \subseteq U$, and there exists $x \in X$ such that $V[x] \in \mathcal{F}$. Then $V[x] \times V[x] \subseteq V \circ V \subseteq U$. Hence $\mathcal{F}$ satisfies definition 2.13.

Conversely, if $\mathcal{F}$ satisfies definition 2.13, then given $U \in \mathcal{U}$ there exists $F \in \mathcal{F}$ such that $F \times F \subseteq U$. For $x \in F$, then $F \subseteq U[x]$ and so $U[x] \subseteq \mathcal{F}$.

However, definitions 2.13 and 2.14 are not equivalent on a quasi-uniform space while 2.13 does imply 2.14.

It is a well known result that if $(X,\mathcal{U})$ is a uniform space and $\mathcal{F}$ is any convergent filter on X then $\mathcal{F}$ is cauchy. This result need not hold on a quasi-uniform space with the usual definition of a cauchy filter. However the result does hold using definition 2.14.

2.16 <u>Lemma</u>. If $(X,\mathcal{U})$ is a quasi-uniform space and $\mathcal{F}$ is any convergent filter on X then $\mathcal{F}$ is cauchy.

Proof: Let $U \in \mathcal{U}$. Let $\mathcal{F}$ converge to $x \in X$. Then $U[x] \in \mathcal{F}$ for all $U \in \mathcal{U}$.

The following is a theorem of Sieber and Pervin [39].

2.17 Theorem. Let f be a uniformly continuous mapping of the quasi-uniform space $(X,\mathcal{U})$ into the quasi-uniform space $(Y,\mathcal{V})$. If $\mathcal{F}$ is a cauchy filter on $(X,\mathcal{U})$ then the set $f\ (F)$, for $F \in \mathcal{F}$, is a cauchy filter base on $(Y,\mathcal{V})$.

Proof: Clearly the collection $f\ (F)$, for $F \in \mathcal{F}$ is a base for some filter $\mathcal{F}^*$ on $(Y,\mathcal{V})$. Let $V \in \mathcal{V}$. By uniform continuity, there exists $U \in \mathcal{U}$ such that $(x,y) \in U$ implies $(f(x), f(y)) \in V$. Since $\mathcal{F}$ is cauchy, there exists a point $x \in X$ such that $U[X] \in \mathcal{F}$, and so $V[f(x)] \in f(\mathcal{F}) \subset \subset \mathcal{F}^*$. Hence $\mathcal{F}^*$ is cauchy.

Therefore if $\mathcal{F}$ is a convergent cauchy filter on the quasi-uniform space $(X,\mathcal{U})$ and f is a uniformly continuous mapping from $(X,\mathcal{U})$ to $(Y,\mathcal{V})$ then the image of $\mathcal{F}$ converges on $(Y,\mathcal{V})$.

Let f be a function from the quasi-uniform space $(X,\mathcal{U})$ to the quasi-uniform space $(Y,\mathcal{V})$, then we can define a map f_2 from

Proof: Let $f \in \Omega$, and $\mathcal{F}$ converge to $x \in X$. Then
$f(x) \in \mathcal{F}'$ for all $x \in X$.

The following is a theorem of Gilbert and Patton [8]:

4.17 Theorem: Let Ω be a uniformly equicontinuous subset of the quasi-uniform space $(X,\mathcal{U})$ into the quasi-uniform space $(Y,\mathcal{V})$. If $\mathcal{F}$ is a Cauchy filter on Ω, then for each $x \in X$, $\mathcal{F}(x)$ is a Cauchy filter base on $(Y,\mathcal{V})$.

Proof: Consider the collection $\{F(x) : F \in \mathcal{F}\}$. It is a base for some filter $\mathcal{F}^*$ on $(Y,\mathcal{V})$. Let $V \in \mathcal{V}$. By uniform equicontinuity, there exists $U \in \mathcal{U}$ such that $(x,y) \in U$ implies $(f(x), f(y)) \in V$ for all $f \in \Omega$. There exists $x \in X$ such that $U(x) \in \mathcal{F}$, and so $\mathcal{F}(x) \subset V(f(x))$, $x \in X$. Hence $\mathcal{F}^*$ is Cauchy.

Therefore if Ω is a uniformly equicontinuous family of functions from a space $(X,\mathcal{U})$ into a uniformly complete space $(Y,\mathcal{V})$ then the image of $\mathcal{F}$ converges in $(Y,\mathcal{V})$.

Let $\mathcal{F}$ be a function from the quasi-uniform space $(X,\mathcal{U})$ to the quasi-uniform space $(Y,\mathcal{V})$, that is the family $\mathcal{V}_{\Omega}$, then

$X \times X$ to $Y \times Y$ be putting $f_2(x,y) = (f(x), f(y))$. Using this definition for f_2 we can show that the primage of a cauchy filter is cauchy.

2.18 Lemma. If $f: X \to Y$ is an onto function where $(X,\mathcal{U})$ and $(Y,\mathcal{V})$ are quasi-uniform spaces, and if $\mathcal{F}$ is a cauchy filter on $(Y,\mathcal{V})$, then $f^{-1}(\mathcal{F})$ is a cauchy filter on X relative to $f_2^{-1}(\mathcal{V}$.

2.19 Lemma. Let $(X,\mathcal{U})$ be a uniform space and let $\mathcal{F}$ be a cauchy filter on X, then $cl\,\mathcal{F} = \lim \mathcal{F}$.

Proof: Let $U \in \mathcal{U}$ and $V \in \mathcal{V}$ such that $V \circ V \subseteq U$. Since $\mathcal{F}$ is cauchy, for some $y \in X$ we have $V[y] \in \mathcal{F}$. Let x be a cluster point of $\mathcal{F}$. Then $V[x] \cap V[y] \neq \phi$, and so $(x,a) \in V$ for some $a \in F$. By $V[y] \times V[y] \subseteq V$, we have $(a,b) \in V$ for every $b \in F$ and so $(x,b) \in V \circ V \subseteq U$. Therefore $V[y] \subseteq V \circ V[x] \subseteq U[x]$. Hence x is a limit point of $\mathcal{F}$.

We were able to obtain a necessary condition for a filter, on a topological space (X,T) with the Pervin quasi-uniformity to be a cauchy filter. The Pervin quasi-uniformity was introduced by Pervin [34] in 1963. A subbasis for the Pervin quasi-uniformity P, is the family of all sets $S_G = G \times G \cup (X - G) \times X$ where $G \in T$.

2.20 Theorem. Let (X,T) be a topological space and $G \in T$ such that $G \not\subseteq \bigcup_{i=1}^{n} T_i$ for all $T_i \in T$ such that $T_i \neq G$. Let $\mathcal{F}$ be any filter on X. If $G \in \mathcal{F}$ then $\mathcal{F}$ is a cauchy filter when $X \times X$ is given the Pervin quasi-uniformity.

Proof: We know $M \subseteq X \times X$ is a member of $\mathcal{P}$ if and only if $M \supseteq S_{H_1} \cap \ldots \cap S_{H_n}$ where each H_i is some member of T and $S_{H_i} = H_i \times H_i \cup (X - H_i) \times X$.

If $G = H_i \neq X$ for some i, then we can always pick $x \in G$ such that $x \notin \bigcup_{i=1}^{n} H_i$. Fix this x.

Consider $B_1 = S_G \cap S_{H_i} \cap \ldots S_{H_m}$. Clearly $(x,y) \in B_1$ for all $y \in G$. Then B_1 is a basis element for $\mathcal{P}$ and $G \subseteq B_1[x]$. Therefore $B_1[x] \in \mathcal{F}$.

Similarly, if $B_2 = S_{H_i} \cap \ldots \cap S_{H_m}$, there exists $x \in G \ni x \notin \bigcup_{i=1}^{n} H_i$ and $G \subseteq B_2[x]$. Since $G \subseteq U[x]$ for some $x \in G$ and all B in the basis for $\mathcal{P}$, hence $G \subseteq U[x]$ for all $U \in \mathcal{P}$ and some $x \in G$. Hence $\mathcal{F}$ is a cauchy filter.

The previous theorem gives an easy construction for presenting a cauchy filter which is not an O-filter.

Under what conditions is every O-filter cauchy? The following theorem which is a sharper form of a result of Sieber and Pervin [39] provides the answer.

2.21 Theorem. In a quasi-uniform space the following are equivalent:

(i) $(X,\mathcal{U})$ is precompact

(ii) Every O-filter on $(X,\mathcal{U})$ is a cauchy filter.

(iii) Every ultrafilter on $(X,\mathcal{U})$ is a cauchy filter.

Remark: A quasi-uniform space $(X,\mathcal{U})$ is precompact if given any $U \in \mathcal{U}$ there exists a finite number of elements $x_1, x_2, \ldots, x_n$ in X such that $X \subseteq U[x_1] \cup \ldots \cup U[x_n]$.

Proof: (i) $\Rightarrow$ (ii). Let $\mathcal{U}$ be an open base for the quasi-uniformity on X and let U be an arbitrary member of $\mathcal{U}$. Since X is precompact there exists a finite set $\{x_1, \ldots, x_n\}$ in X such that $X = \bigcup_{i=1}^{n} U[x_i]$, since each $U[x_i]$ is open, some $U[x_K] \in \mathcal{F}$ where $1 \leq K \leq n$ and $\mathcal{F}$ is an arbitrary O-filter on X. Thus $\mathcal{F}$ is cauchy.

(ii) $\Rightarrow$ (iii). Since every ultrafilter is an O-filter, (iii) follows from (ii).

(iii) $\Rightarrow$ (i). If X is not precompact then there exists $U \in \mathcal{U}$ such that $X - U[A] \neq \emptyset$ for all finite subsets A of X. The collection $\mathcal{B} = \{ X - U[A] \mid A \text{ is finite} \}$ form a filter base on X. Let $\mathcal{F}$ be an ultrafilter containing $\mathcal{U}$. By hypothesis $\mathcal{F}$ is cauchy and therefore for some $x \in X$, $U[x] \in \mathcal{F}$. But this contradicts the fact that $X - U[x] \in \mathcal{B}$. Hence X is precompact.

Ross [37] has proven the following:

2.22 <u>Theorem.</u> Let X be a separable uniformizable space. If $\mathcal{F}$ is a filter on X without a cluster point, then there exists a uniform structure, compatible with the topology for X, for which $\mathcal{F}$ is a cauchy filter.

Proof: (of course, a separated uniformizable space means that the topology associated with the uniform structure $\mathcal{U}$ is a Hausdorff topology).

Let $a \in X$ and U_a' an arbitrary neighbourhood of a. Since a is not a cluster point of $\mathcal{F}$, there exists $F \in \mathcal{F}$ such that $a \notin F$. Pick $W_a \subset V_a'$ such that $a \in W_a$ and $W_a \subseteq F^c$. Let f be a continuous function in particular $f(x) = 1$ for all $x \in F$. Let $\epsilon > 0$ be arbitrary. Since (x) is continuous, to every $x \in X$ there corresponds U_x^ϵ such that $y \in V_x^\epsilon$ implies

$| f(x) - f(y) | < \epsilon$. Consider V_x^ϵ to be the set of points y such that $| f(x) - f(y) | < \epsilon$.

Put $V_\epsilon = \bigcup_{x \in X} V_x^\epsilon \times V_x^\epsilon$.

Let $b \in F$. Then $F(b) = 1$ and $f(y) = 1$ for $y \in F$. Hence $F \subset V_b^\epsilon$, i.e. $F \times F \subseteq V_\epsilon$. It can be easily shown that the family V_ϵ $(\epsilon > 0)$, generate a family of entourages for the set X . The V_ϵ (depending upon ϵ and a) form a subbasis for a uniformity $\mathcal{U}$. $\exists$ $V \in \mathcal{U}$, therefore,

$$V = V_{\epsilon_1}^{a_1} \cap V_{\epsilon_2}^{a_2} \cap \ldots \cap V_{\epsilon_n}^{a_n} .$$

This uniformity will introduce the original topology on X .

Finally, we will show that $\mathcal{F}$ is a cauchy filter.

$$F^{a_1} \times F^{a_1} \subset V_{\epsilon_1}^{a_1} \ldots\ldots F^{a_n} \times F^{a_n} \subseteq V_{\epsilon_n}^{a_n}$$

hence the set $F = F^{a_1} \cap F^{a_2} \cap \ldots \cap F^{a_n}$ is such that

$$F \times F \subset V$$

Lemma 2.19 says that in a uniform space a cauchy filter converges to its cluster points. However, this may not be true in a quasi-uniform space. But if every filter on (X,U) has a cluster point then every cauchy filter converges.

2.23 <u>Theorem</u>. In a compact quasi-uniform space $(X,\mathcal{U})$ every cauchy filter converges.

Proof: Let $\mathcal{F}$ be a non-convergent cauchy filter in the compact quasi-uniform space $(X,\mathcal{U})$. For every $x \in X$, x is not a limit point of $\mathcal{F}$ and so there exists some $U_x \in \mathcal{U}$ such that $U_x[x] \notin \mathcal{F}$. We know there exists $V_x \in \mathcal{U}$ such that $V_x \circ V_x \subseteq U_x$. Then $\{ V_x[x] : x \in X \}$ is a covering of X and since X is compact there exists a finite subcover $V_{x_1}[x_1] \ldots$ $\ldots V_{x_n}[x_n]$. Let $V = \bigcap_{i=1}^{n} V_{x_i}$. Since $\mathcal{F}$ is cauchy, there exists a point $Z \in X$ such that $V[Z] \in \mathcal{F}$. But $Z \in V_{x_i}[x_i]$ for some i and so $V[Z] \subseteq U_{x_i}[x_i]$ which implies $U_{x_i}[x_i] \in \mathcal{F}$. A contradiction. Hence $\mathcal{F}$ converges.

2.24 <u>Theorem</u>. If (X,T) is a topological space such that every open set is also closed, then every convergent filter is an O-filter.

Proof: Let A be an open set in X and $\mathcal{F}$ a convergent filter on X. For some $x \in X$, every nbhd N_x of x is a member of $\mathcal{F}$. But x has a neighbourhood contained in A or A^c, depending upon whether $x \in A$ or $x \in A^c$.

Hence A or $A^c \in \mathcal{F}$.

4.69 Theorem. In a compact quasi-uniform space (X,U) every [illegible] filter converges.

Proof: Let $\mathcal{F}$ be a [illegible] filter in the compact quasi-uniform space (X,U). For every $x \in X$ [illegible] points of $\mathcal{F}$ [illegible] $U_x \in U$ [illegible] $U_x(x) \notin \mathcal{F}$. [illegible] $V_x \in U$ [illegible] $V_x \circ V_x \subseteq U_x$. Then $\{ V_x(x) : x \in X \}$ [illegible] and since X is compact there exists a finite subcover $\{ V_{x_i}(x_i)$, [illegible] $V_{x_n}(x_n) \}$. Let $Y = \bigcap_{i=1}^{n} V_{x_i}$. Since $\mathcal{F}$ is [illegible] there exists a point $F \in \mathcal{F}$ such that [illegible] $x \in V_{x_i}(x_i)$ for some i and so $F(x) \subseteq U_{x_i}(x_i)$ which implies $U_{x_i}(x_i) \in \mathcal{F}$, a contradiction. Hence $\mathcal{F}$ converges.

4.70 Theorem. If (X,U) is [illegible] [illegible] with [illegible] and is also [illegible] then every [illegible] filter [illegible]

Proof: Let [illegible] and [illegible] U [illegible] For some $x \in X$ [illegible] U [illegible] [illegible] [illegible] $x \in X$

[illegible] $\mathcal{F}$ [illegible]

The last two theorems gives immediately the following relationship between cauchy filters and O-filters.

2.25 <u>Corollary</u>. If (X,U) is a compact quasi uniform space such that every open set is also closed then every cauchy filter is an O-filter.

An hausdorff space which satisfies corollary 2.25 will be finite, since every singleton is closed and hence open and the space is discrete. Also a T_0 space which satisfies the condition of the corollary is hausdorff since every open set is closed. Therefore any non finite space which has the properties of corollary 2.25 cannot satisfy any separation axiom.

So far we have only considered cauchy filters. As would be expected, a cauchy net is its counter part in the theory of nets.

2.26 <u>Definition</u>. A net (x_a) $(a \in A)$ in a uniform space $(X,\mathcal{U})$ is said to be a <u>cauchy net</u> if given $U \in \mathcal{U}$ there exists $N \in A$ such that for all $m , n \geq N$ $(x_n , x_m) \in U$.

2.27 <u>Lemma</u>. A net (x_a) $(a \in A)$ on a uniform space (X,U) is a cauchy net if and only if the associated filter base is a cauchy filter base.

The proof of this lemma follows directly from the definition and is omitted. We shall return to cauchy filters later when we discuss a completion for a uniform space.

C. REGULAR FILTERS

Banaschewski [3] defines an open filter as follows:

2.28 Definition. A filter $\mathfrak{F}$ on a topological space (X,T) which has a base $\mathcal{B}$ consisting of open sets will be called an open filter.

$\mathcal{B}$ is called an open filter base. An application of Zorn's lemma shows that every open filter base is contained in a maximal open filter. Obreanu [31] in a paper on open filters discusses five well-known properties concerning maximal open filters.

2.29 Definition. A filter $\mathfrak{F}$ on a topological space (X,T) which has a base $\mathcal{B}$ consisting of closed sets will be called a closed filter.

$\mathcal{B}$ is then called a closed filter base.

2.30 Definition. A filter base will be called regular if it is open and is equivalent to a closed filter base.

Regular filters were introduced by Sorgenfry and Berri [8] in order to characterize minimal regular spaces. We shall return to open, closed and regular filters later in our study of minimal topological spaces.

D. COMPLETELY REGULAR FILTERS

The literature contains two distinct definitions for a completely regular filter. Bourbaki [13] gives the following:

2.31 Definition. A filter $\mathfrak{F}$ on a completely regular space is said to be a completely regular filter if there exists an open base $\mathcal{B}$ for $\mathfrak{F}$ such that for every $B \in \mathcal{B}$ there exists $A \in \mathcal{B}$, with $A \subset B$, and a continuous function f on X, such that $f(A) = 0$ and $f(X-B) = 1$.

Notation: We shall refer to a filter satisfying definition 2.31 as a B C R filter.

Alfsen and Fenstad [1] have given the following:

2.32 Definition. A filter $\mathcal{F}$ on a uniform space $(X,\mathcal{U})$ is a completely regular filter if the collection of sets $V[F]$ form a base for $\mathcal{F}$, where $V \in \mathcal{U}$ and $F \in \mathcal{F}$.

Notation: We shall refer to a filter which satisfies definition 2.32 as a CR filter.

We shall show that a filter on a uniform space which satisfies definition 2.32 also satisfies definition 2.31. First we need a lemma.

2.33 Lemma. If $(X,\mathcal{U})$ is a uniform space and F is an open subset of X then there exists an open subset A of X such that $A \subseteq F$ and a continuous function f on X such that $f(A) = 0$ and $f(X-F) = 1$.

Proof: Let $x \in F$. Choose $V \in \mathcal{V}$, an open base for the uniformity $\mathcal{U}$, such that $V(x) \subseteq F$. We will show that $V(x)$ is the required subset, A. The proof is now exactly the same as theorem 11. 2.2 Pervin [33], where V plays the role of $\mathcal{U}_0$. f plays a role similar to f with the exception that $f(y)$ is defined to be zero for all $y \in V(x)$. Then the required result follows.

Alfsen and Fuglede [1] have given the following.

2.38 Definition. A filter $\mathcal{F}$ on a uniform space [illegible] is a completely regular filter, if the intersection of each [illegible] form a base for $\mathcal{F}$, where [illegible] and [illegible]

Remark. We shall refer to a filter which satisfies definition 2.38 as a [illegible] filter.

We shall show that a filter on a [illegible] space which satisfies definition 2.38 also satisfies definition 2.31. First we prove a lemma.

2.39 Lemma. If [illegible] is a [illegible] space and [illegible] then there exists an open subset [illegible] such that [illegible] and a continuous function [illegible] such that [illegible] and $f(F) = 1$.

Proof. Let $x \in F$. Choose [illegible] , we can [illegible] uniformity [illegible] such that [illegible] . We will show that [illegible] is the required subset [illegible] . The proof is now exactly the same as theorem [illegible] [illegible] , where [illegible] [illegible] with the assumption that [illegible] defined to be [illegible] for all [illegible] . Then the required result follows.

2.34 <u>Theorem.</u> Let $\mathcal{F}$ be a CR filter on a uniform space $(X,\mathcal{U})$ then $\mathcal{F}$ is a BCR filter.

Proof: Let $\mathcal{V}$ be an open base for the uniformity $\mathcal{U}$. Let $F \in \mathcal{F}$ then $V[F]$ is an open member of the base for $\mathcal{U}$. Then there exists $F' \in \mathcal{F}$ and $V' \in \mathcal{V}$ such that $V'[F'] \subseteq V[F]$ where $V'[F']$ is an open member of the base for $\mathcal{F}$. Then by lemma 2.33, with x replaced by F', there exists a continuous function f such that $f(V'(F')) = 0$ and $f(X - V[F]) = 1$ and $\mathcal{F}$ is a C R filter.

We are unable to decide as to whether the converse to theorem 2.34 holds. However it is conjectured that the converse is false.

2.35 <u>Definition.</u> A (B)CR filter is <u>maximal</u> if and only if it is not properly contained in any other (B)CR filter.

Using Zorn's lemma and it follows that every (B)CR filter is contained in a maximal (B)CR filter.

The following theorem is an analogue of lemma 1.30

2.36 <u>Theorem.</u> (Bourbaki) A BCR filter, $\mathcal{F}$, on a completely regular space X is a maximal BCR filter on X if and only if for

for every pair A, B of open sets of X and a continuous function f on X into $[0,1]$ which satisfies the condition $A \subset B$, $f(A) = 0$ and $f(X-B) = 1$, then either $B \in \mathcal{F}$ or $\mathcal{F}$ has a set which does not meet A.

Proof: Let $\mathcal{B}$ be an open base for the BCR filter $\mathcal{F}$. Assume that $F \cap A \neq \emptyset$ for every $F \in \mathcal{F}$. We will show that in this case $\{F \cap B\}$, for $F \in \mathcal{F}$, is a base for a completely regular filter which is a refinment of $\mathcal{F}$.

Let F and G be members of $\mathcal{B}$ such that $G \subseteq F$ and let f_F be a continuous function on X such that $f_F(G) = 0$ and $f_F(X - F) = 1$. Define a function g on X as follows: $g = f_A$ for all $x \in F$ such that $x \notin B$, and $g = f_F$ otherwise. Then g is continuous at every point of X since f_A and f are both continuous, hence g is continuous on X. Then $f(B \cap G) = 0$ and $f(X - F \cap B) = 1$. Clearly $\{F \cap B\}$ is a refinment of $\mathcal{B}$. Contradiction the maximality of $\mathcal{F}$. Therefore the necessary part of the hypothesis is established.

Conversely, if $\mathcal{B}$ is not a maximal CR filter base, then there exists a maximal CR filter base $\mathcal{G}$ which is a refinment of $\mathcal{F}$. Let $G \in \mathcal{G}$ such that $G \notin \mathcal{B}$. Clearly there exists $G_1 \in \mathcal{G}$ and f continuous on X such that $f(G_1) = 0$ and $f(X - G) = 1$, where $G_1 \cap F = \emptyset$ for some $F \in \mathcal{B}$. Contradicting the hypothesis.

It follows from lemma 2.33 that if X is a completely regular space, then the neighbourhood filter of $x \in X$ is a maximal BCR filter.

Wagner [45] has shown the following:

2.37 <u>Lemma.</u> Every ultrafilter on a completely regular space X is stronger than a maximal BCR filter on X.

Proof: Let U be an ultrafilter on X. Let $\mathcal{F}$ be a BCR filter on X which is a refinment of ever BCR filter contained in U. Assume $\mathcal{F}$ is not a maximal BCR filter. Then by theorem 2.36 there exists open sets A and B such that $A \subset B$ and a continuous function f from X to $[0,1]$ such that $f(A) = 0$, $f(X - B) = 1$ and $B \notin \mathcal{F}$ and every set of $\mathcal{F}$ meets A. Let $\mathcal{G}$ be a filter on X with base $\mathcal{B}$, where each $B(t) \in \mathcal{B}$ is of the form $B(t) = \{ x \in X \mid f(x) < t \}$ where $0 < t < 1$. Then $\mathcal{G}$ is a BCR filter and $B \in \mathcal{G}$. Therefore $\mathcal{G}$ is not properly contained in U. Let t_0 be such that $B(t_0) \in \mathcal{G}$ and $B(t_0) \notin U$. Then $X - B(t_0) \in U$. Let $\mathcal{H}$ be a filter on X with base $\mathcal{C}$, where each $C \in \mathcal{C}$ is of the form $C(t) = \{ x \in X \mid f(x) > t \}$ and since $C(t) \supset X - B(t_0)$, where $0 < t < t_0$. $\mathcal{H} \subset U$. Hence $\mathcal{H} \subset \mathcal{F}$ and $C(t_0/2) \in G$ which does not meet A. Hence a contradiction.

We now confine our attention to the work done on the completely regular filters of Fenstad and Alfsen.

Samuel [38] defined the envelope of a filter in a manner similar to the completely regular filters of Alfsen and Fenstad.

2.37 Definition. Let $\mathcal{F}$ be a filter on a uniform space $(X,\mathcal{U})$. The collection $V[F]$ for $V \in \mathcal{U}$ and $F \in \mathcal{F}$ is the envelope of $\mathcal{F}$.

Let $\mathcal{F}^*$ denote the envelope of a filter $\mathcal{F}$.

2.38 Lemma. Let $\mathcal{F}$ be any filter then the envelope $\mathcal{F}^*$ of $\mathcal{F}$ is a base for a CR filter.

Proof: Let $F^* \in \mathcal{F}^*$. Then $F^* = V[F]$ for some $V \in \mathcal{U}$ and $F \in \mathcal{F}$. Let W be an arbitrary member of $\mathcal{U}$. Then since $W(V(F)) = V \circ W[F]$, $W(V(F)) \in \mathcal{F}^*$.

2.39 Lemma. Let $\mathcal{F}^* \subseteq \mathcal{F}$. If $\mathcal{G}$ is any other CR filter such that $\mathcal{G} \subseteq \mathcal{F}$ then $\mathcal{G} \subseteq \mathcal{F}^*$ (i.e. $\mathcal{F}^*$ is the finest CR filter coarser than $\mathcal{F}$).

Proof: Let $G \in \mathcal{G}$. Since the collection $V(G)$ is a base for $\mathcal{G}$, then there exists some $G' \in \mathcal{G}$ such that $V(G') \subseteq G$. since $G' \in \mathcal{F}$, $V(G') \in \mathcal{F}^*$. Hence $G \in \mathcal{F}^*$.

Let A be a fixed set of X. Then denote by $\mathcal{F}_A$ the filter generated by the collection of all set B such that $V[A] \subseteq B$ for some $V \in \mathcal{U}$.

2.40 <u>Lemma.</u> A CR filter $\mathcal{F}$ converges to x if and only if $\mathcal{F} = \mathcal{F}_x$.

Proof: Let $\mathcal{F}$ converge to x. Then clearly $\mathcal{F}_x \subseteq \mathcal{F}$. But it is also easily seen that $\mathcal{F}_x$ is a maximal CR filter. Hence $\mathcal{F}_x = \mathcal{F}$.

Conversely, let $\mathcal{F}$ be a CR filter such that $\mathcal{F}_x = \mathcal{F}$. Let N_x be a neighbourhood of x, then $N_x \in \mathcal{F}_x$ and hence converges to x.

The following lemma is an analogue of lemma 1.30, a well known result for ultrafilters by H. Cartan. First we need a definition.

2.41 <u>Definition.</u> A finite cover $\{A_i\}_{i=1}^n$ of a uniform space is a p-covering if there exists a finite covering $\{B_i\}_{i=1}^n$ such that $V[A_i] \subset B_i$ for some $V \in \mathcal{U}$.

Proof. Let $U \in \mathcal{U}$. Using the hypothesis (iii) in a base for $\mathfrak{B}$, then there exists some $V \in \mathfrak{B}$ such that $V(\frac{1}{2}B) \subset U$, since $V \in \mathfrak{B}$, $V(B) \in \mathcal{U}$, hence $U \in \mathcal{U}$.

Let B be a fixed set of L, then denote by $\mathfrak{B}_B$ the filter generated by the collection of all sets B such that $V(B) \subseteq B$ for some $V \in \mathfrak{B}$.

2.40 Lemma : A filter $\mathfrak{F}$ converges to [illegible] if and only if $\mathfrak{F} \supset \mathfrak{B}_{[illegible]}$.

Proof. Let $\mathfrak{F}$ converge to [illegible]. Then clearly $\mathfrak{B}_{[illegible]} \subset \mathfrak{F}$. And it is also easily seen that $\mathfrak{B}_{[illegible]}$ is a maximal [illegible] filter. Hence $\mathfrak{B}_{[illegible]} = \mathfrak{F}$.

Conversely, let $\mathfrak{F}$ be a filter such that $\mathfrak{B}_{[illegible]} \subset \mathfrak{F}$. Let $\mathfrak{B}_{\alpha}$ be a neighbourhood of [illegible], then $\mathfrak{B}_{\alpha} \in \mathfrak{F}$, so $\mathfrak{F}$ converges to [illegible].

The following lemma is an analogue of Lemma 1.[illegible], a well known result for [illegible] filters by [illegible]. First we need a definition.

2.41 Definition. A finite cover $\{U_i\}_{i=1}^{n}$ of a uniform space [illegible] is generated by [illegible] if there exists a finite covering $\{A_i\}_{i=1}^{n}$ such that $V(A_i) \subset U_i$ for some $V \in \mathfrak{B}$.

2.42 <u>Lemma.</u> A CR filter $\mathcal{F}$ is maximal if and only if for $H \in \mathcal{F}$ and for every p-covering $\{A_i\}_{i=1}^{n}$ of H, at least one of the sets $A_i \in \mathcal{F}$.

Proof: Let $\mathcal{F}$ be a maximal CR filter, $H \in \mathcal{F}$ and let $\{A_i\}_{i=1}^{n}$ be a p-covering of H. Proceed by induction, clearly the lemma holds when $n = 1$. Assume lemma holds for $n - 1$. Let $A_n' = \bigcup_{i=1}^{n-1} A_i$.

Then $\{A_n', A_n\}$ is a p-covering of H.

Assume $A_n' \notin \mathcal{F}$ and $A_n \notin \mathcal{F}$. The family of sets $F \cap A_n^c$ for $F \in \mathcal{F}$ is a filter on X. Then the envelope of this filter, $\mathcal{G}$, is a CR filter. Since $\mathcal{G}$ is a refinment of $\mathcal{F}$, $\mathcal{G} = \mathcal{F}$.

Since $\{A_n', A_n\}$ is a p-covering of H, there exists sets B, B' such that $H \subseteq B \cup B'$, $V[B] \subset A_n$ and $V'[B'] \subseteq A_n'$ for B and $V' \in \mathcal{U}$.

Since $H \in \mathcal{F}$ and $H \cap A_n^c \subset A_n \subset B \subset V'[B'] \subseteq A_n'$. But $\mathcal{G}$ is CR, hence $A_n' \in \mathcal{G}$. Therefore $A_n' \in \mathcal{F}$ contrary to assumption.

Conversely, assume $\mathcal{F}$ is a CR filter with the property stated in the lemma but $\mathcal{F}$ is not maximal. Let $\mathcal{G}$ be a maximal CR filter finer than $\mathcal{F}$. Let $G \in \mathcal{G}$ such that $G \notin \mathcal{F}$. Since $\mathcal{G}$ is CR $V[H] \subseteq G$ for some $V \in \mathcal{U}$, and $H \in \mathcal{G}$. Hence $\{H^c, G\}$ is a p-covering. Since $G \notin \mathcal{F}$, therefore $H^c \in \mathcal{F}$. This implies $H^c \in \mathcal{G}$. A contradiction.

2.43 Lemma. Let $(X,\mathcal{U})$ be a quasi uniform space and $\mathcal{F}$ is a CR convergent filter on X. Then $\mathcal{F}$ is a maximal CR filter.

Proof: Let $\{A_i\}_{i=1}^{n}$ be an arbitrary p-covering. Then for some p-covering $\{B_i\}_{i=1}^{n}$ and some $V \in \mathcal{U}$, $V[B_i] \subseteq A_i$. $x \in B_i$ for some i. This implies $x \in V[B_i] \subseteq A_i$. Hence $V[x] \subseteq A_i$. But $v[x] \in \mathcal{F}$, hence $A_i \in \mathcal{F}$ and $\mathcal{F}$ is maximal.

2.44 Corollary. In a compact quasi uniform space (X,U) every CR cauchy filter is maximal CR filter.

Alfsen and Fenstad have shown:

2.45 Lemma. Let $\mathcal{F}$ be a CR cauchy filter on a uniform space (X,U), then $\mathcal{F}$ is a maximal CR filter.

Proof: Let $\{A_i\}_{i=1}^{n}$ be an arbitrary p-covering of X. Then $V = \bigcup_{i=1}^{n} A_i \times A_i$ is a member of $\mathcal{U}$ (The family $\mathcal{V}$ of all such entourages, V, constructed from the collection of all p-coverings of X form the coarsest uniformity on X. See [2]). Then $F \times F \subseteq V$ for some $F \in \mathcal{F}$. Clearly $F \subset A_i$ for some i and $A_i \in \mathcal{F}$. Hence $\mathcal{F}$ is maximal.

2.46 __Definition.__ Two cauchy filter bases are __equivalent__ if their intersection is a cauchy filter base.

2.47 __Lemma.__ Let $\mathcal{F}$ be a cauchy filter then envelope, $\mathcal{F}^*$, of $\mathcal{F}$ is also cauchy.

Proof: Let V be an entourage on X. Choose an entourage V_1 such that $V_1 \circ V_1 \circ V_1 \subset V$. Let $F \in \mathcal{F}$ such that $F \times F \subseteq V_1$. Then $V_1[F] \times V_1[F] \subseteq V_1 \circ V_1 \circ V_1 \subset V$. Hence $\mathcal{F}^*$ is cauchy.

2.48 Lemma. Every equivalence class of cauchy filters contains exactly one (maximal) CR filter.

Proof: From lemma 2.43 it is clear that every equivalence class of cauchy filters contains at least one CR filter $\mathcal{F}^*$. Assume $\mathcal{F}'$ is a second CR filter equivalent to $\mathcal{F}^*$. Let $\mathcal{G} = \mathcal{F}' \cap \mathcal{F}^*$. Clearly $\mathcal{G}$ is cauchy and also CR, hence by lemma 2.41 maximal CR, since $\mathcal{G} \subset \mathcal{F}'$ and $\mathcal{G} \subset \mathcal{F}^*$, $\mathcal{G} = \mathcal{F}' = \mathcal{F}^*$.

The next important result of Alfsen and Fenstad gives the relationship between maximal CR filters and cauchy filters. Before stating this result we need a necessary and sufficient condition for a separated uniform space to be compact.

2.49 <u>Lemma.</u> A separated uniform space $(X,\mathcal{U})$ is compact if and only if every maximal regular field is convergent.

Proof: If X is compact and $\mathcal{F}$ is any maximal CR filter than $\mathcal{F}$ has a cluster point x. $\mathcal{F}$ converges to this cluster point which is unique, since X is Hausdorff.

Conversely, if $\mathcal{F}$ is an ultrafilter on X, then $\mathcal{F}^*$, the envelope of $\mathcal{F}$, is a maximal CR filter. Since $\mathcal{F}^*$ converges $\mathcal{F}$ converges and X is compact.

2.50 <u>Theorem.</u> A separated uniform space $(X,\mathcal{U})$ is precompact if and only if every maximal CR filter on X is cauchy.

Proof: Let f be the function which embedds X into its completion $\bar{X}$. Let $\mathcal{F}$ be a CR filter on $\bar{X}$. Then, since X is dense in $\bar{X}$ and $\mathcal{U}$ is the restriction of $\bar{\mathcal{U}}$ to X, $f^{-1}(\mathcal{F})$ is a maximal CR filter on X. Hence $f^{-1}(\mathcal{F})$ is cauchy. Therefore $\mathcal{F}$ is cauchy in $\bar{X}$. Since $\mathcal{F}$ converges, by lemma 2.45 $\bar{X}$ is compact. Hence X is precompact being a dense subset of a compact uniform space.

Conversely, let $\{A_i\}_{i=1}^{n}$ be a p-covering of X. Then $V = \cup\{A_i \times A_i\}$ is an entourage on X. If $\mathcal{F}$ is a maximal CR filter then by lemma 2.38 for each V there is some i such that $A_i \in \mathcal{F}$. But $A_i \times A_i \subseteq V$. Hence $\mathcal{F}$ is cauchy.

CHAPTER III

SOME CHARACTERIZATIONS USING NETS AND FILTERS

A. COMPACTNESS

It can be shown that a topological space (X,T) is compact if and only if every ultra filter on X converges. Since the convergence of nets and filters are equivalent ideas - a space is compact if and only if every universal net converges. Bartle [4] and Kelley [25] both characterize compactness using nets, while Gaal [22] uses filters. We shall also, in a manner similar to Robertson and Franklin [36], give an easy proof of the Tychonoff theorem using O-filters.

With the usual definition of compactness, an application of De Morgan's laws show that a topological space (X,T) is compact if and only if every family of closed subsets of X with the finite intersection property has non-empty intersection.

3.1 Theorem. A topological space (X,T) is compact if and only if every filter base $\mathfrak{F}$ on X has a cluster point.

Proof: If X is compact and $\mathfrak{F} = \{B_a \mid a \in A\}$ is a filter base on X then $\bigcap_{i=1}^{n} \overline{B}_{a_i} \neq \emptyset$ and hence $\bigcap_{a\in A} \overline{B}_a \neq \emptyset$.

Hence, by the definition of cluster point, $\mathcal{F}$ has a cluster point.

Conversely, assume X is not compact. Then there exists a family $\mathcal{F}$ of X such that $\bigcap_{i=1}^{n} \overline{F}_{a_i} \neq \emptyset$ for all finite subfamilies of $\mathcal{F}$, while $\cap \overline{F}_a = \emptyset$, and $\mathcal{F}$ is a filter base without a cluster point.

3.2 <u>Theorem.</u> If X is a topological space, the following are equivalent:

(i) X is compact.

(ii) Every O-filter on X converges.

(iii) Every ultra filter converges.

Proof: (i) $\Rightarrow$ (ii). Since X is compact every O-filter has a cluster point which in turn is a limit point by lemma 2.5.

(ii) $\Rightarrow$ (iii). Since every ultrafilter is an O-filter.

(iii) $\Rightarrow$ (i). If every ultra filter on X converges then every filter on X must have a cluster point. Hence X is compact.

We are now in a position to present an easy proof of the Tychonoff product theorem.

3.3 <u>Theorem.</u> The topological product of any family of compact spaces is compact.

Proof: Let $\mathcal{F}$ be an arbitrary O-filter in the product space. Lemma 2.6 tells us that the projection of $\mathcal{F}$ into each coordinate space is also an O-filter, and hence convergent. This implies $\mathcal{F}$ is convergent. Hence the product space is compact.

B. A TOPOLOGY DEFINED IN TERMS OF NETS

3.4 **Definition.** Given a space X , a topology on X is a family of subsets of X satisfying the following conditions:

(i) $X , \emptyset \in T$,

(ii) union of an arbitrary number of members of T , is a member of T ,

(iii) Finite intersection of members of T , is a member of T .

(X,T) is then a topological space and the members of T are said to be open sets.

Birkoff [10] has shown that a topology may be defined on an arbitrary space X via nets. In terms of nets, what do we mean by an open set?

3.5 **Definition.** A set $G \subseteq X$ is open if and only if no net outside of G converges to a point of G .

We will also say that if (x_a) $(a \in A)$ is any net on X then the following two properties hold.

(i) If $(x_a)(a \in A) = x$ for every $a \in A$ then x_a converges to x.

(ii) If (x_a) $(a \in A)$ converges to x and B is any cofinal subset of A then (x_b) $(b \in B)$ also converges to x.

3.6 Theorem. If X is any collection of points, in which it is decided which nets converge to which points, then X is a topological space under the criterion of 3.5.

Proof: We have to show that the open set of definition 3.5 satisfy the conditions of definition 3.4.

(i) Clearly X, $\emptyset$ are open.

(ii) If S is a union of open sets, G, satisfying 3.5, does S satisfy 3.5? But this is clearly the case, since a net outside of S is certainly outside of each G and hence the net does not converge to a point of S.

(iii) Let (x_a) $(a \in A)$ be any net in X and let B be any subset of A which is not cofinal, then clearly $A - B$ is cofinal. We want to show that if S and T are open in the sense of definition 3.5 so is $S \cap T$. Any net (x_a) $(a \in A)$ outside of $S \cap T$ must contain a cofinal subnet outside of S or else one outside of T. This cofinal subnet can converge to no point of $S \cap T$, hence neither can (x_a) $(a \in A)$. Therefore $S \cap T$ is open.

Kowalsky [27] defines a topology on a set X in terms of neighbourhood filters. He first considers the lattice of filters, $X(F)$, on X and defines a mapping τ from X to $X(F)$. τ is said to be a topology for X if the following conditions are satisfied:

(i) $P(x) \leq \tau(x)$ for all $x \in X$ where

$P(x) = \{ M \in X \mid M \subset X \text{ and } x \in M \}$

(ii) $\bigwedge\limits_{U \in \tau_p} \bigvee\limits_{q \in U} \tau(q) \leq \tau(p)$.

where $\wedge$, $\vee$ is the meet and join, respectively, in the lattice sense.

C. T_1, T_2, AND T_3 SEPARATION AXIOMS

The following characterization of T_1, and T_3 space are due to Kowalsky [27].

Notation: Let X be any space and $p \in X$. By $P(p)$ we shall means the collection of all $M \subseteq X$ such that $p \in M$.

3.7 Theorem. Let (X,T) be a topological space in which the neighbourhood filter base $N(x)$ for each x is open. Then the following are equivalent:

(i) $N(x) \subseteq P(p)$ implies $p = x$.

(ii) Given two distinct points p, x in X there exists an open neighbourhood $U \in N(x)$ such that $p \notin U$.

(iii) The intersection of all $U \in N(x) = \{x\}$.

(iv) Every subset of X consisting of a single point is closed.

Proof: (i) $\Rightarrow$ (ii). Suppose $p \in U$ for all $U \in N(x)$. Then $P(p) \supseteq N(x)$ and by (i), $p = x$. Hence there exists $U \in N(x)$ such that $p \notin U$. Since $N(x)$ has an open base we may choose U open.

(ii) $\Rightarrow$ (iii). Since $N(x) \subseteq P(x)$ we have $x \in \cap \{ U : U \in N(x) \}$. By (ii) we have $p \notin \cap \{ U : U \in N(x) \}$ for all $p \neq x$.

(iii) $\Rightarrow$ (iv). If $p \neq x$, then $p \notin \cap \{ U : U \quad N(x) \}$ and hence for some neighbourhood U of x $p \notin U$. This implies $x \in \overline{\{p\}}$. Since this is true for all $x \in X$, $\overline{\{p\}} = p$.

(iv) $\Rightarrow$ (i). If $N(x) \subseteq P(p)$ this implies $U \cap R \neq \emptyset$ for every $U \in N(x)$ and every $R \in P(p)$. Hence $x \in \overline{\{p\}} = \{p\}$ and $p = x$.

The following theorem by Birkoff [10] is a characterization of Hausdorff spaces.

3.8 <u>Theorem</u>. Let (X,T) be any topological space and for each $x \in X$ let $N(x)$ be the open nbhd filter of x. Then the following are equivalent:

(i) If x and y are any two distinct points of X, then there exists $U \in N(x)$ and $V \in N(y)$ such that $U \cap V = \emptyset$.

(ii) If $\mathcal{F}$ is any convergent filter on X then $\mathcal{F}$ has at most one limit point.

Proof: (i) $\Longrightarrow$ (ii). Let $\mathcal{F}$ be a filter on X and assume $\mathcal{F}$ converges to both x and y where $x \neq y$. This implies if $U \in N(x)$ and $V \in N(y)$ then U and V are members of $\mathcal{F}$. But by (i) this is impossible for all U and V. Hence $\mathcal{F}$ converges to at most one point.

(ii) $\Longrightarrow$ (i). If (i) does not hold. Then there exists points x and y in X such that $x \neq y$ and for every $U \in N(x)$ and for every $V \in N(y)$, $U \cap V \neq \emptyset$. This implies $\{U \cap V\}$ is a filter base on X which converges to both x and y.

Kowalsky [27] uses the following definition for a regular space:

3.9 <u>Definition</u>. A space (X,T) is regular if and only if there exists a set of closed neighbouhoods $\mathcal{B}(x)$ for every $x \in X$ such that $\mathcal{B}(x)$ is a basis for $N(x)$.

3.10 Theorem. The following are equivalent:

(i) (X,T) is regular

(ii) Given any $U \in N(x)$, there exists a neighbourhood $V \in N(x)$ such that $\overline{V} \subseteq U$.

(iii) Given any closed set A and any point $x \notin A$, there exists open sets U and V such that $x \in U$ and $A \subseteq V$ and $U \cap V = \emptyset$.

Proof: (i) $\Longrightarrow$ (ii) Let $U \in N(x)$. Then there exists $B \in \mathcal{B}(x)$ such that $B \subseteq U$, since B is a closed neighbourhood of X there exists an open neighbourhood V of X such that $V \subseteq \overline{V} \subseteq B \subseteq U$.

(ii) $\Longrightarrow$ (i) For every $U \in N(x)$, there exists a closed $V \in N(x)$ such that $V \subseteq U$. Hence the collection of all such V's is a closed basis for $N(x)$.

(ii) $\Longrightarrow$ (iii) Since A is closed, A^c is open and $x \in A^c$. Then $A^c \in N(x)$ and there exists $U \in N(x)$ such that $\overline{U} \subset A^c$. Hence $V = U^c$ is open and $A \subset V$. Therefore $U \cap V = \emptyset$ and $x \in U$ and $A \subset V$.

(iii) $\Longrightarrow$ (ii) Let $U \in N(x)$. Then $\text{Int}\,(U) \in N(x)$. $(\text{Int } U)^c$ is a closed set not containing x and hence there exists open sets V and W such that $x \in V$ and $(\text{Int } U)^c \subseteq W$ and $V \cap W = \emptyset$. Since $x \in V$ we have $V \in N(x)$. In addition $\overline{V} \subseteq W^c \subseteq \text{Int } U \subseteq U$.

D. MINIMAL TOPOLOGICAL SPACES

Berri and Sorgenfry [8] defined a minimal topological space as follows:

3.11 Definition. Let X be a space and T be a topology on X . (X,T) is minimal P if (X,T) has property P and for any other topology T_1 on X such that $T_1 \subset T$ then (X,T_1) does not have property P .

A topological space (X,T) will be said to satisfy condition S if the following two properties hold:

(a) Every open filter base has a cluster point.

(b) If an open filter base has a unique cluster point, then it converges to this point.

3.12 Lemma. A Hausdorff space which satisfies (b) also satisfies (a).

Proof: Assume there exists an open filter base $\mathcal{F}$ which has no cluster point. Choose some $p \in X$. Let $\mathcal{B}$ be the filter base of open neighbourhoods of p . Let $\mathcal{G} = \{ V \cup F \mid V \in \mathcal{B} , F \in \mathcal{F} \}$. Clearly p is the only cluster point of $\mathcal{G}$ and $\mathcal{G}$ converges to p by hypothesis. But $\mathcal{G} \subseteq \mathcal{F}$. Hence $\mathcal{F}$ converges to p . Hence p is a cluster point $\mathcal{F}$.

3.13 Lemma. If a Hausdorff space (X,T) is minimal then it satisfies condition (b).

Proof: Let $\mathfrak{F}$ be an open filter base with a unique cluster point p and assume that $\mathfrak{F}$ does not converge to p. We will now construct a topology T' which is weaker than T. For each $x \in X$ let $\mathcal{U}(x)$ be the open filter base of x. Let $\mathcal{U}'(x) = \mathcal{U}(x)$ if $x \neq p$. If $x = p$ let $\mathcal{U}'(p) = \{ U \cup F \mid F \in \mathfrak{F}, U \in \mathcal{U}(x) \}$. Define T' to be the topology on X such that $\mathcal{U}'(x)$ is an open filter base at x for each $x \in X$. Then T' is strictly weaker than T since there is a $U \in \mathcal{U}(p)$ which contains no set of $\mathcal{U}'(p)$. T' is Hausdorff. Hence (X,T) is not minimal Hausdorff.

3.14 Lemma. If (X,T) is a Hausdorff space satisfying (b) then (X,T) is minimal.

Proof: Assume T is not minimal Hausdorff. Let T' be smaller than T. Let $\mathcal{U}(x)$ and $\mathcal{U}'(x)$ be the open sets containing x for each $x \in X$ in T and T', respectively. In the T' topology x is a cluster point of $\mathcal{U}'(x)$. Hence, by hypothesis, the filter base generated by $\mathcal{U}'(x)$ converges in the T topology. Hence $\mathcal{U}(x) \subseteq \mathcal{U}'(x)$ and T is weaker than T'. Hence T is minimal.

Hence the following theorem follows:

3.15 <u>Theorem.</u> A Hausdorff space (X,T) is minimal Hausdorff if and only if it satisfies condition S .

Proof: follows from 3.12, 3.13 and 3.14.

3.16 <u>Theorem.</u> If (X,T) is a compact Hausdorff space then T is minimal Hausdorff.

Proof: Let T_0 be weaker than T and assume T_0 is Hausdorff. Let G be an open set in T . Then G^c is closed in T and hence compact. Since $T_0 \subset T$, this implies G^c is closed in T_0 . Hence $G \in T_0$ and $T \subset T_0$. Hence T is minimal Hausdorff.

The converse to the above lemma does not hold. Uryshon [44] has given an example of a minimal Hausdorff space which is not compact. Berri and Sorgenfry [8] have also classified characterized minimal regular space using regular filters. We will follow their classification and say that condition R holds on a topological space (X,T) if (X,T) has the following properties:

(α) Every regular filter-base [see Defn. 2.30] which has a unique cluster point is convergent.

(β) Every regular filter base has a cluster point.

3.17 Lemma. A regular space (X,T) which satisfies (α) also satisfies (β).

Proof: The proof is similar to lemma 3.12 with open filter base replace by regular filter base.

3.18 Theorem. In order that a regular space (X,T) be minimal it is necessary and sufficient that is satisfy property (α).

The proof of this theorem is omitted because of its similarity with lemmas 3.13 and 3.14.

If (X,T) is any non minimal Hausdorff space the following lemma gives a method for constructing a topology for X which is strictly small than T.

3.19 Lemma. Let (X,T) be a Hausdorff space. Let $\mathcal{F}$ be an open filter base on X which does not have a cluster point (or if $\mathcal{F}$ has a cluster point it fails to converge to it). Fix some point $p \in X$, (if $\mathcal{F}$ has a cluster point let it be p). Define the following family of filter bases on X.

$$\mathcal{B}(x) = \begin{cases} \mathcal{V}(x) \text{, the } T\text{-neighbourhood system of } X \text{ if } x \neq p ; \\ \{ U \cup F \mid U \text{ is an open neighbourhood of } X \text{ and } F \in \mathcal{F} \} \\ \quad \text{for } x = p . \end{cases}$$

Then for each $x \in X$, $\mathcal{B}(x)$ determines a neighbourhood system for a Hausdorff topology T^* such that $T^* \subset T$.

Proof: It is clear that T^* is a topology on X . For $x , y \in X$ such that $x \neq p$ and $y \neq p$ it is clear that x and y have disjoint neighbourhoods.

Let $y = p$ and $x \neq p$. There exists an open nbhds U and V of x and p , respectively, in T such that $U \cap V = \emptyset$. Since T is Hausdorff and x is not a cluster point of $\mathcal{F}$, there exists an open neighbourhood W of x and an open set $F \in \mathcal{F}$, where $W , F \in T$, such that $W \cap F = \emptyset$. Then $U \cap W$ is an open neighbourhood of x in T^* which does not intersect $V \cup G$, an open neighbourhood of p in T^* . Clearly $T^* \subset T$.

The above lemma holds on a non minimal regular space (X,T) if we choose the open filter base $\mathcal{F}$ such that it is equivalent to a closed filter base $\mathcal{G}$. This lemma is stated by Berri [9].

Berri [9] has also shown:

3.20 Theorem. A minimal Tychonoff space (X,T) is compact.

Proof: Assume X is not compact. Let $\beta(X)$ the Stone-Cech compactification of X . Since X is not compact choose $a \in \beta(X) - X$. Let $\mathcal{F}$ be the open neighbourhood filter of a in $\beta(X)$. a is the

only cluster point of $\mathcal{F}$. Let $\mathcal{F}^* = \{ F \cap X \mid F \in \mathcal{F} \}$. Then $\mathcal{F}^*$ is an open filter base on X which does not have a cluster point in X . Select and fix any point $p \in X$. Then construct the topology T^* on X as defined in lemma 3.19. Clearly $T^* \subset T$ and (X,T^*) is regular. We will show that (X,T^*) is a Tychonoff space.

Let Ω , Ω^*, Ω^{**} be the family of real valued continuous functions from (X,T) , (X,T^*) and $\beta(X)$, respectively, onto the unit interval. Let $b \in X$ and F a closed subset of X in the T^* topology such that $b \notin F$.

Case I: If $b \neq p$. It can be shown that there exists $f \in \Omega^*$ such that $f(b) = 0$ and $f(F) = 1$, where f is also a member of Ω .

Case II: $b = p$. Consider the closure of F in $\beta(X)$. Then we can find an $f^{**} \in \Omega$ such that $f^{**} \in \Omega^*$ and $f^{**}(b) = 0$ and $f^{**}(F) = 1$.

3.21 <u>Theorem</u>. All minimal norm spaces are compact.

Berri has also proven this theorem, the proof is very similar to theorem 3.20 and it is therefore omitted.

E. COMPLETION OF A UNIFORM SPACE

It is a classic result that every metric space has a completion. In the 1920's and 1930's many mathematicians were considering the problem of completing an arbitrary topological space. Birkoff [11] talked about completions in terms of closure strictly in a philisophical nature. As a matter of fact, he showed that completness is usually associated with a closure property which determines its uniqueness. As a consequence of Birkoff's paper on Moore-Smith convergence Graves [23] felt that the completion of a topological space should be studied in terms of neighbourhood systems. He defined a 'fundamental directed set (x_α) ' and defined the space to be complete if and only if every "fundamental directed set" has a limit point in the Moore-Smith sense. His definition of a 'fundamental directed set' is what has become to be known as a cauchy net.

Cohen [19], [20] defined a uniformity in terms of the neighbourhood system and defined a space to be complete if and only if every cauchy net has a limit point. With these definitions he showed that every uniform space has a completion.

Weil [47] defined a uniformity as a lighter condition than that used by Cohen (this was shown by Cohen [21]). He defined a cauchy filter and considered the space to be complete if and only if every

cauchy filter has a limit point. He also proved that a topological space is uniformizable if and only if it is completely regular.

3.22 <u>Definition</u>. A uniform space $(X,\mathcal{U})$ is <u>complete</u> if and only if every cauchy filter on X converges.

This is clearly equivalent to defining $(X,\mathcal{U})$ to be complete if and only if every cauchy net on X converges.

3.23 <u>Definition</u>. The pair $(X,\mathcal{U})$ and $(Y,\mathcal{V})$ are called <u>uniformly isomorphic</u> if and only if there is a one-to-one correspondence $f\colon X \to Y$ between the elements of X and Y such that the map f_2 from $X \times X$ to $Y \times Y$, where $f_2(x_1, x_2) = (f(x_1), f(x_2))$, yields a one-one correspondence between the elements of $\mathcal{U}$ and $\mathcal{V}$.

3.24 <u>Definition</u>. A pair $(\overline{X},\overline{U})$ is called a <u>completion</u> of the pair (X,U) if $\overline{U}$ is a complete uniform structure on $\overline{X}$ such that

(i) $\tilde{X}$ is dense in $\overline{X}$ relative to the uniform topology induced by $\overline{U}$, and

(ii) The pair (X,U) is uniformly isomorphic to $\tilde{X}$ and the relativization of $\overline{U}$ to $\tilde{X}$.

Murdeshwar and Naimpally [30] have given the following:

3.25 <u>Definition</u>. A quasi uniform structure (X,Q) is <u>complete</u> if and only if every cauchy filter on X has a cluster point.

This is equivalent to definition 3.22 since on a uniform space a cauchy filter converges to each of its cluster points.

As mentioned earlier there are many different methods employed to show that every uniform space has a completion. The question as to whether every quasi-uniform space has a completion remains unsolved. Kelley [24] and Isbel [26] show that every uniform space has a completion by showing that every uniform space is uniformly isometric to a subspace of the product of pseudo-metric spaces. Then they show the following lemmas:

(i) A closed subspace of a complete uniform space is complete.

(ii) Every metric space can be embedded in a product of metric spaces.

(iii) A product of complete metric spaces is complete.

From this it follows that every uniform space has a completion.

Gaal [22] and Bourbaki [12] have constructed a completion for an arbitrary uniform space using cauchy filters with arbitrary small open sets. A collection of these filters are considered equivalent if they contain

the same open sets. Denote the equivalence classes by $\xi, \eta \ldots$. The neighbourhood filters of x and y are denoted by $\tilde{x}, \tilde{y} \ldots$ etc. Then the collection of ξ, η are the points of $\bar{X}$, while $\tilde{x}, \tilde{y} \ldots$ are the points of $\tilde{X}$. From the uniformity $\mathcal{U}$ on X a uniformity $\bar{\mathcal{U}}$ is constructed from $\mathcal{U}$ as follows: If $\bar{U} \in \bar{\mathcal{U}}$ then

$$\bar{U} = \{(\xi, \eta) \mid O \times Q \subseteq U \text{ for some } O \in \xi \text{ and } Q \in \eta\}$$

where U is a symmetric open member of $\mathcal{U}$. Then $(\bar{X}, \bar{\mathcal{U}})$ is complete and $\tilde{X}$ is dense in $\bar{X}$.

Robertson and Robertson [35] have constructed a completion of a Hausdorff uniform space using minimal cauchy filters (a cauchy filter on a space X is minimal if there exists no strictly coarser cauchy filter on X) .

Alfsen and Fenstad [1] formed a completion of a uniform space using completely regular filters.

Recently Parek and Naimpally [32] constructed a completion for a uniform space in which they defined a cauchy filter as in 2.14. Their construction is relatively simple, but ingenious, and we present the main ideas in their argument.

Let X be a non-empty set and $\mathcal{U}$ be a symmetric base for a uniformity on X.

3.26 <u>Lemma.</u> Let $\bar{A} = X$. If every filter base on A which is cauchy in X has a cluster point in X, then X is complete.

Proof: Let $\mathcal{F}$ be a cauchy filter on X.

Then $\mathcal{F}_A = \{ A \cap U[F] \mid U \in \mathcal{U}, F \in \mathcal{F} \}$ is a filter base on X. By hypothesis $\mathcal{F}_A$ has a cluster point $x \in X$. It is clear that $\mathcal{F}_A$ is a cauchy filter base on X, since $\mathcal{F}$ is cauchy. Therefore $\mathcal{F}_A$ converges to x. We must show that $\mathcal{F}$ converges. Let $U[x]$ be an arbitrary neighbourhood of x. There exists $V \in \mathcal{U}$ such that $V \circ V \subseteq U$. Clearly $V[x] \cap (A \cap V[F]) \neq \emptyset$ for all $F \in \mathcal{F}$. This implies there exists $b \in F$ such that $b \in V[F] \subseteq V \circ V[x] \subseteq U[x]$. Therefore $U[x] \cap F \neq \emptyset$ for all $F \in \mathcal{F}$ and hence x, being a cluster point, is a limit point of $\mathcal{F}$.

3.27 <u>Definition.</u> Let $\bar{X}$ be the class of all cauchy filter bases on X.

3.28 <u>Definition.</u> Let $\bar{U} = \{ (\mathcal{F}, \mathcal{G}) \in \bar{X} \times \bar{X} \mid$ there exists $F \in \mathcal{F}$, $G \in \mathcal{G}$ such that $A \subset U[b]$ for each $b \in G \}$, for each $U \in \mathcal{U}$.

3.29 <u>Lemma.</u> $\bar{\mathcal{U}}$, the collection of all $\bar{U}$ as defined above, is a symmetric base for a uniformity for $\bar{X}$.

Proof: (i) $\Delta \subseteq \bar{U}$ for each $\bar{U} \in \bar{\mathcal{U}}$: for given any $U \in \mathcal{U}$ there exists $V \in \mathcal{U}$ such that $V \circ V \subset U$, then for every $\mathcal{F} \in \bar{X}$, there is $x \in X$, $F \in \mathcal{F}$ such that $A \subseteq V[x]$. So for every $a \in F$, $F \subseteq V[x] \subset V \circ V[a] \subset U[a]$ which implies $(\mathcal{F}, \mathcal{F}) \in \bar{U}$.

(ii) Similarly it can be shown that if $W \subset V \cap U$ then $\bar{W} \subset \bar{U} \cap \bar{V}$.

(iii) Also for each $\bar{U} \in \bar{\mathcal{U}}$, $\bar{U} = \bar{U}^{-1}$.

(iv) Given $\bar{U} \in \bar{\mathcal{U}}$, to show there exists $\bar{V} \in \bar{\mathcal{U}}$ such that $\bar{V} \circ \bar{V} \subseteq \bar{U}$.

For every $U \in \mathcal{U}$ there exists $V \in \mathcal{U}$ such that $V \circ V \subseteq U$. Let $(\mathcal{F}, \mathcal{G}) \in \bar{V}$. Then there exists $(\mathcal{F}, \mathcal{H}) \in \bar{V}$ and $(\mathcal{H}, \mathcal{G}) \in \bar{V}$, where $\mathcal{H} \in \bar{X}$ such that for some $F \in \mathcal{F}$ and $G \in \mathcal{G}$ there exists H_1 and $H_2 \in \mathcal{H}$ such that $F \subseteq V[h_1]$ for $h_1 \in H_1$ and $H_2 \subseteq V[g]$ for $g \in G$. Since $\mathcal{H}$ is a filter, $H = H_1 \cap H_2 \neq \emptyset$ and $F \subseteq V \circ V[g] \subset U[g]$ for each $g \in G$ and $(\mathcal{F}, \mathcal{G}) \in \bar{U}$.

3.30 <u>Definition</u>. For each $x \in X$, let $\tilde{x}$ denote the set of all supersets of X containing x. Let $\tilde{X}$ denote the collection of all these subsets of X.

Since each $\tilde{x}$ is a cauchy filter it is clear that $\tilde{X} \subseteq \bar{X}$.

If φ is the mapping from X to $\tilde{X}$ such that $\varphi(x) = \tilde{x}$ then both φ and φ^{-1} are uniformly continuous and hence X and

$\tilde{X}$ are uniformly isomorphic.

3.31. Lemma. $\tilde{X}$ is dense in $\bar{X}$.

Proof: Let $\mathcal{F} \in \bar{X}$ and $\bar{U} \in \bar{\mathcal{U}}$. Then for some $x \in X$ and some $A \in \mathcal{F}$, $A \subseteq U[x]$. Then $(\mathcal{F}, \tilde{x}) \in \bar{U}$ and $\tilde{X}$ is dense in $\bar{X}$.

3.32 Lemma. Every filter base in $\tilde{X}$ which is cauchy in $\bar{X}$ converges in $\bar{X}$.

Proof: Since $\tilde{X}$ is isomorphic to X, every cauchy filter in $\tilde{X}$ is of the form $\varphi(\mathcal{F})$ where $\mathcal{F} \in \bar{X}$. We must show that $\varphi(\mathcal{F})$ converges in $\bar{X}$. Let $\bar{U} \in \bar{\mathcal{U}}$ then there exists $\bar{V} \in \bar{\mathcal{U}}$ such that $\bar{V} \circ \bar{V} \subseteq \bar{U}$. Since $\mathcal{F}$ is cauchy on X, there exists $F \in \mathcal{F}$ and $x \in X$ such that $F \subseteq V[x]$. Since V is symmetric $F \subseteq V \circ V[a] \subseteq U[a]$ for every $a \in F$. Then $(\mathcal{F}, \varphi(a)) \in \bar{U}$, which implies $\varphi(a) \in \bar{U}[\mathcal{F}]$ and $\varphi(\mathcal{F})$ converges to $\mathcal{F}$.

By lemma 3.26 $\bar{X}$ is complete.

Combining lemmas 3.26 to 3.32 and we have the following:

3.33 Theorem. Every uniform space $(X,\mathcal{U})$ is uniformly isomorphic to a dense subset of a complete uniform space $(\bar{X},\bar{\mathcal{U}})$.

F. CONVERGENCE ON FILTERS

Convergence on filters was introduced by Brace [14] as a replacement for simple uniform convergence of Bourbaki. In this section we shall deal with the main ideas of convergence on filters as presented by Brace and show how it leads to a necessary and sufficient condition for a filter of functions continuous at a point x to converge to a function which is also continuous at x.

Notation: $G(X;Y)$ denotes a family of functions f whose common domain is a set X with values in the Hausdorff uniform space $(Y,\mathcal{U})$.

3.33 Definition. A net of function (f_a) $(a \in A)$ on a set X to a uniform space $(Y,\mathcal{U})$ converges (pointwise) to a function f_0 if given $x_0 \in X$ and $U \in \mathcal{U}$ there exists $a_1 \in A$ such that for all $a \geq a_1$ $f_a(x) \in U[f_0(x)]$ for all $x \in X$.

3.34 Definition. A net of function (f_a) $(a \in A)$ on a set X to a uniform space $(Y,\mathcal{U})$ converges uniformly to f_0 if given $U \in \mathcal{U}$ there exists $a_1 \in A$ such that for all $a \geq a_1$ $f_a(x) \in U[f_0(x)]$ for all $x \in X$.

3.35 Definition. A sequence of function (f_a) $(a \in A)$ from a set X to a uniform space $(Y,\mathcal{U})$ is simply uniformly convergent at x_0 of X to a function f_0 if

(i) it converges pointwise on a nbhd of x_0.

(ii) for every $U \in \mathcal{U}$ and positive integer n_0 there exists an integer $n \geq n_0$ such that $f_n(x) \in U[f_0(x)]$ for all $x \in X$.

It is clear that if X is a compact topological space and (f_a) $(a \in A)$ is a sequence of function on X which is simply uniformly convergent to f_0 then (f_a) $(a \in A)$ converges uniformly to f_0.

3.35 <u>Definition</u>. Let $\mathcal{G}$ be a filter of function from a set X to a uniform space $(Y, \mathcal{U})$. $\mathcal{G}$ <u>converges to a function f_0 on a filter $\mathcal{F}$</u> on X if for every $U \in \mathcal{U}$ there is a $G \in \mathcal{G}$ such that for each f in G there is a F_f in $\mathcal{F}$ with the property that $(f(x), f_0(x)) \in U$ for all $x \in F_f$.

$\mathcal{G}$ is then said to converge on the filter $\mathcal{F}$.

The fundamental relationship between convergence on filters and continuous functions was given by Brace in the following:

3.36 <u>Theorem</u>. Let $\mathcal{G}$ be a filter of function from X to a uniform space $(Y, \mathcal{U})$ where every $g \in \mathcal{G}$ is continuous at $x_0 \in X$. The filter $\mathcal{G}$ converges at x_0 to a function f_0 which is continuous at x_0 if and only if it converges to f_0 on the filter of neighbourhoods of x_0.

Proof: Let $\mathcal{G}$ converge at x_0 to a function f_0. Then given $U \in \mathcal{U}$ there exists $G \in \mathcal{G}$ such that for all $g \in G$ $g(x_0) \in U[\, f_0(x_0)\,]$. Since each g and f_0 are continuous,

for each g there exists V_g (a nbhd of x_0) such that $(g(x), f_0(x)) \in U$ for all $x \in V_g$. Hence $\mathcal{G}$ converges to f_0 on the filter nbhd of x_0.

Conversely, let U be an arbitrary entourage on Y. Then there exists $V \in \mathcal{U}$ such that $V \circ V \circ V \subseteq U$. By hypothesis there exists $G \in \mathcal{G}$ such that for every $g \in G$ there exists V_g of x_0 such that $g(x) \in V[f_0(x)]$ for all $x \in V_f$. Also since $\mathcal{G}$ converges to f_0 at x_0, $g(x_0) \in V[f_0(x_0)]$. Thus $(f_0(x), f_0(x_0))$ is in U for all x in V_g and f_0 is continuous at x_0.

Brace [15] has also constructed a <u>Topology of convergence on a Filter</u> for the set of functions from X to the uniform space $(Y, \ )$. Let $\mathcal{U}$ be a base for the uniformity on Y. Let $\mathcal{F}$ be a fixed filter on X. Let $\mathcal{V}$ denote the collection of all sets of the form $V(U,\mathcal{F})$, where

$V(U,\mathcal{F}) = \{ (f,g) \mid$ where $F_{fg} \in \mathcal{F}$ such that $(f(x),g(x)) \in U$ for all $x \in F_{fg} \}$ for all $U \in \mathcal{U}$.

3.37 <u>Lemma</u>. The set $V(U,\mathcal{F})$ form a base for a uniformity on $G(X,Y)$.

Proof: Let $U \in \mathcal{U}$ and $V(U,\mathcal{F}) \in \mathcal{V}$.

(1) Since the diagonal is contained in each $U \in \mathcal{U}$, each $V \in \mathcal{V}$ also contains the diagonal in $G(X,Y)$.

(2) Since $U^{-1} \in \mathcal{U}$ for each $U \in \mathcal{U}$ if $V(U,\mathcal{F}) \in \mathcal{V}$ then so is $V^{-1}(U,\mathcal{F})$.

(3) If $V(U,\mathcal{F}) \in \mathcal{V}$, then there exist $U' \in \mathcal{U}$ such that $U' \circ U' \subseteq U$ and we will show that $V(U',\mathcal{F}) \circ V(U',\mathcal{F}) \subseteq V(U,\mathcal{F})$ since $V(U',\mathcal{F}) = \{(f,g) \mid$ for some $F'_{fg} \in \mathcal{F}$, $(f(x),g(x)) \in U'$ for all $x \in F'_{fg}\}$ and since $U' \circ U' \subseteq U$ it follows $V(U',\mathcal{F}) \circ V(U',\mathcal{F}) \subseteq V(U,\mathcal{F})$.

(4) Let $V(U',\mathcal{F})$, $V(U^2,\mathcal{F}) \in \mathcal{V}$. If (f,g) belongs to their intersection then there exists F_{fg}, $F_{fg} \in \mathcal{F}$ such that $(f(x),g(x)) \in V'$ for all $x \in F'_{fg}$ and $(f(x),g(x)) \in V^2$ for all $x \in F^2_{fg}$. This implies $(f(x), g(x)) \in V' \cap V^2$ for all x in $F'_{fg} \cap F^2_{fg}$.

Hence $V(U',\mathcal{F}) \cap V(U^2,\mathcal{F}) \subseteq V(U' \cap U^2, \mathcal{F})$. Similarly it can be shown that the inclusion holds in the opposite direction.

Hence $\mathcal{V}$ is a base for a uniformity of $G(X,Y)$.

Let G (X,Y) denote the collection of all filters on $G(X,Y)$ which converge on the filter $\mathcal{F}$.

3.38 <u>Lemma.</u> Any collection of filters $\mathcal{H}$ (in $G(X,Y)$) converges to f_0 on the filter $\mathcal{F}$ if and only if it converges to f_0 on the topology associated with $\mathcal{F}$.

Proof; Let $\mathcal{G}$ be a filter in $\mathcal{H}$ converging to f_0 on the filter $\mathcal{F}$. Then for each entourage U of Y there exists $G \in \mathcal{G}$ such that for each $g \in G$ there exists an F_g in $\mathcal{F}$ with the property that $(g(x),f_0(x)) \in U$ for all $x \in F_g$. Then $\mathcal{G}$ converges to f_0 relative to the topology associated with the uniformity.

Conversely, let $\mathcal{G}$ be a filter which converges to f_0 relative to the topology associated with the uniformity. This implies given V an entourage on Y , $(g(x),f(x)) \in V$ for all $x \in X$ and for all $g \in G \in \mathcal{G}$. Then $\mathcal{G}$ converges to f_0 on the filter $\mathcal{F}$.

G. <u>QUASI-UNIFORM CONVERGENCE</u>

Quasi-uniform convergence was introduced for a sequence of function by Arzela in 1883. Bartle [7] has stated a theorem of Arzela's which gives a necessary and sufficient condition for a net of continuous functions from a topological space to a metric space to converge to a continuous function. This theorem is important functional analysis and we shall generalize it to the case when the range space is uniformizable.

We have already defined what is meant by a family of functions (f_α) converging point wise, [3.33], and uniformly, [3.34], to a function f_0 on a space X.

3.38 <u>Definition</u>. A net (f_a) $(a \in A)$ of functions on an arbitrary set X is said to converge to f_0 <u>quasi-uniformly on X</u>, if $f_a(x)$ converges to $f_0(x)$ for all $x \in X$ and if for every $U \in \mathcal{U}$ and $a_0 \in A$ there exists a finite subfamily $\alpha_1, \ldots \alpha_n \geq a_0$ in A such that for each $x \in X$ at least one of the following is true

$$f_{\alpha_i}(x) \in U[\, f_0(x)\,],\ i = 1, \ldots, n.$$

3.39 <u>Lemma</u>. If a net of functions (f_a) $(a \in A)$ converges on X, and if some subnet converges quasi-uniformly on X, then the net converges quasi-uniformly on X.

The proof of the above lemma is clear and is omitted.

The importance of quasi-uniform convergence stems from the fact that it is a necessary and sufficient condition for a net of continuous functions to converge to a continuous function.

3.40 <u>Lemma</u>. If a net of coninuous functions (f_a) $(a \in A)$ on a topological space X to a uniform space Y converges to a continuous function f_0, then the convergence is quasi-uniform on every compact subset of X.

Proof: Let $\mathcal{U}$ be an open base for the uniformity on Y. Since f_0 is continuous on X then given $U \in \mathcal{U}$, $a \in A$ and $y \in X$, there is an $a(y) \geq a_0$ such that $f_{a(y)} \in U[f_0(y)]$. Let $G_y = \{ x \in X \mid f_{a(y)}(x) \in U[f_0(x)] \}$. Since f_0 and f are both continuous G_y is open and $y \in G_y$. If K is any compact subset of X, only a finite number of sets G_{y_i}, $y_i \in K$ and $i = 1 \ldots n$ are required to cover K. The indecies $a(y_i) \ldots i = 1 \ldots n$ then satisfy the requirements of definition 3.38.

3.41 <u>Lemma</u>. Let (f_a) $(a \in A)$ be a net of functions on a space X to uniform space Y. If (f_a) $(a \in A)$ converges to f_0 quasi-uniformly on $B \subseteq X$ then f_0 is continuous on B.

Proof: Let $U \in \mathcal{U}$, then there exist $V \in \mathcal{U}$ such that $V \circ V \circ V \subseteq U$. We want to show that f_0 is continuous on B. Let $x_0 \in B$ then since (f_a) $(a \in A)$ converges to f_0, there exists $a_0 \in A$ such that for $a \geq a_0$ $f_a(x_0) \in V[f_0(x_0)]$. Select $a_1, \ldots, a_n \geq a_0$ to satisfy definition 3.38. Let $N_i = \{ x \in B \mid f_{a_i}(x) \in V[f_{a_i}(x_a)] \}$. The set N_i is open and contains x_0. Hence $N = \bigcap_{i=1}^{n} N_i$ is open and contains x_0. For every $x \in N \cap B$ we have $(f_0(x), f_0(x_0)) \in U$ since $V \circ V \circ V \subseteq U$ and $(f_0(x), f_{a_i}(x)) \in V$, $(f_{a_i}(x), f_{a_i}(x_0)) \in V$

and $(f_{a_i}(x_0)\ ,\ f_0(x_0)) \in V$ for proper choice of i . Hence f_0 is continuous on B .

3.42 Theorem. On a compact topological space, the limit of a net of continuous functions is continuous if and only if the convergence is quasi-uniform.

The proof follows from the preceding lemmas.

H. FILTER-SPACES

Filter-spaces were introduced by Wagner [45] . He used them to construct a compactification for a completely regular topological space. He claims that "these filter-spaces include, for a completely regular space all its compactification".

Let X be a T_0 space and $\mathcal{T}$ a topology on X . Let $\mathcal{E} = \{ N(x) \mid x \in X \}$ where $N(x)$ denotes the neighbourhood filter of $x \in X$. We now define a space φ , such that φ is a collection of filters on X , as follows: (i) $\varphi \supset \mathcal{E}$ and (ii) every filter on φ is open. For every $U \in \mathcal{T}$ put $r(U) = \{ \mathcal{F} \in \varphi \mid U \in \mathcal{F} \}$.

3.43 <u>Definition.</u> The space φ equipped with the topology γ, where the collection of sets $r(U)$ is a base for γ, is called a <u>filter-space</u>.

The following three Lemmas may then be established:

3.44 <u>Lemma.</u> The mapping $e : X \to \xi$ defined by $e(x) = N(x)$ is a homeomorphism of X into φ.

Proof: Clearly e is one-one. Let G be a basic open set in φ containing $N(x)$. Then $G = \{ \mathcal{F} \in \varphi \mid U \in \mathcal{F} \}$ and U is open in X. Then $e^{-1}(G) = U$. Similarly e^{-1} is continuous.

3.45 <u>Lemma.</u> ξ is dense in φ.

Proof: Let $F \in \varphi$ such that $F \notin \xi$. Let G be an arbitrary basic open set of φ. Then G is of the form $\{ \mathcal{F} \in \varphi \mid U \in \varphi \}$ where U is open in X and hence a neighbourhood of some x, which implies $V(x) \in G$. Hence $\mathcal{F}$ is dense in φ.

If $\mathcal{F}$ is a filter base on X then it is clear that $e(\mathcal{F}) = \{ e(F) \mid F \in \mathcal{F} \}$ is a filter base on φ.

3.46 <u>Lemma.</u> In the space φ, the filter base $e(\mathcal{F})$ converges to $\mathcal{G} \in \varphi$ if and only if $\mathcal{F} \supset \mathcal{G}$.

Proof: Let $U \in \mathcal{G}$. Since $\mathcal{G}$ is an open filter, we may choose U to be open. To show that $U \in \mathcal{F}$. $r(U) = \{\mathcal{H} \in \varphi \mid U \in \mathcal{H}\}$. Since $r(U)$ is an open neighbourhood of $\mathcal{G}$, $r(U) \in \mathcal{F}$. Hence $r(U) \supseteq e(F)$ for some $F \in \mathcal{F}$ such that F is open in X. $e(F) = \{ V(x) \in \varphi \mid R \in V(x) \} \supseteq \{\mathcal{H} \in \varphi \mid U \in \mathcal{H} \}$. Hence $F \subseteq U$ and $U \in \mathcal{F}$.

Conversely, let $\mathcal{F} \supseteq \mathcal{G}$. Then given an open set $G \in \mathcal{G}$, $G \in \mathcal{F}$. Let U be an open set of $\mathcal{G}$, then $r(U)$ is a basis open set in φ which contains $\mathcal{G}$. Since $U \in \mathcal{F}$ and $r(U) \supseteq e(U)$, $e(U) \in e(\mathcal{F})$ and $\mathcal{F}$ converges to $\mathcal{G}$.

3.47 <u>Corollary</u>. In the space φ, the filter base $e(\mathcal{F})$ converges to $V(x)$ if and only if $\mathcal{F}$ converges to x.

The corollary follows immediately from lemma 3.46.

3.48 <u>Lemma</u>. In the filter-space φ, $\mathcal{G} \in \varphi$ is a cluster point of the filter base $e(\mathcal{F})$ if and only if every set of $\mathcal{G}$ meets every set of $\mathcal{F}$.

Proof: Let $U \in \mathcal{G}$, if $r(U) \cap e(F) = \emptyset$, then clearly $F \cap U = \emptyset$.

Conversely, if $F \cap G = \emptyset$ for all $F \in \mathcal{F}$ and $G \in \mathcal{G}$, then $r(G) \cap e(G) \neq \emptyset$ where G and F are open.

3.49 Corollary. In the space φ, $V(x)$ is a cluster point of $e(\mathcal{F})$ if and only if x is a cluster point of $\mathcal{F}$.

Let Φ be an open filter on φ. Since e is a homomorphism from X to a subset of φ, $e^{-1}(\Phi) = \{ e^{-1}(\beta) | \beta \in \Phi \}$ is an open filter base on X.

3.50 Lemma. In the filter space φ, $\mathcal{G} \in \varphi$ is a cluster point of Φ if and only if every set of $\mathcal{G}$ meets every set of $e^{-1}(\Phi)$.

Proof: Let $\mathcal{G}$ be a cluster point of Φ and $r(U)$ be an arbitrary basis open set containing $\mathcal{G}$. If $\beta \in \Phi$, then $r(U) \cap \beta \neq \emptyset$. Let $\mathcal{H} \in r(U) \cap \beta$. Since β is open, there exists a basis open set $r(V)$ such that $\mathcal{H} \in r(V) \subseteq \beta$. Then $r(U) \cap r(V) \neq \emptyset$ and $V \cap U \neq \emptyset$. Since $V \subseteq e^{-1}(\beta)$, $U \cap e^{-1}(\beta) \neq \emptyset$.

Conversely, since for every $G \in \mathcal{G}$ and $\beta \in \Phi$, $G \cap e^{-1}(\beta) \neq \emptyset$, let $x \in G \cap e^{-1}(\beta)$. Hence $V(x) \in r(G)$ and $V(x) \in r(G)$. Hence $\mathcal{G}$ is a cluster point of Φ.

Before stating a necessary and sufficient condition, which is due to Wagner, for a filter space to be compact we need the following result.

3.51 Lemma. Let (X,T) be a regular topological space then X is compact if and only if every open filter base on X has a cluster point in X.

Proof: Clearly, if X is compact every open filter base on X has a cluster point.

Conversely, assume that every open filter base on X has a cluster point. Assume there exists some filter $\mathcal{F}$ on X such that $\mathcal{F}$ does not have a cluster point. Then for every $x \in X$ there exists an open neighbourhood U of x and $G \in \mathcal{F}$ such that $G \cap U = \emptyset$. Since X is regular, there exists an open nbhd V of x such that $\overline{V} \subseteq U$. Then $G \subseteq \overline{V}^{c}$. Let $\mathcal{V}$ be the collection of all such $\overline{V}^{c}$ as x runs over the space X. $\mathcal{V}$ is an open filter base on X and clearly since $V \cap \overline{V}^{c} = \emptyset$, $\mathcal{V}$ does not have a cluster point on X. A contradiction. Hence X is compact.

3.52 Theorem. A filter space φ over a completely regular space X is compact if and only if φ is regular and every ultra filter on X is stronger than at least one filter of φ.

Proof: If the filter-space φ is compact then every filter on has a cluster point. Let U be an ultrafilter on X. Then $e(U)$ is a filter base on φ and has a cluster point $\mathcal{F} \in \varphi$. By lemma 3.48, U is stronger than $\mathcal{F}$.

Conversely, let Φ be an open filter base on φ then $e^{-1}(\Phi)$ is an open filter base on X. Let U be an ultrafilter on X which is a refinment of $e^{-1}(\Phi)$. Let $\mathcal{F}$ be a filter on X such that $\mathcal{F} \in \varphi$ and $\mathcal{F}$ is contained in U. Then by lemma 3.50, $\mathcal{F}$ is a cluster point of Φ. By lemma 3.51, φ is compact.

I. ULTRAFILTER SPACES

Ultrafilter spaces were introduced in 1948 by Samuel [38] and he used these spaces to show that every Hausdorff uniformizable space can be embedded into a compact space.

Consider any set X and let the subsets of X be ordered by inclusion. Let Ω denote the collection of all ultrafilters of X and Ω_A the set of all ultrafilters containing a given subset A of X. Let Ω_A^* denote the complement of Ω_A in Ω. Samuel then defined two topologies T_O and T_F on Ω, which are both equal for our consideration since the complement of every set exists. The topology is therefore defined on Ω by taking the subsets Ω_A as a subbase for closed sets; or what is equivalent to taking Ω_A^* as a subbase for open sets.

3.53 Definition. The space Ω equipped with the topology T is an ultrafilter space.

Since every filter is contained in an ultrafilter and since every filter is generated by the principal filter determined by its elements, we see that the set $\Omega_{\mathcal{F}}$ of all ultrafilters finer than a given filter $\mathcal{F}$ is closed in T . Hence every closed set is a finite union of sets of the form $\Omega_{\mathcal{F}}$.

3.54 <u>Lemma</u>. (Ω, T) is a Hausdorff space.

Proof: Let U and V be any two distinct ultrafilters of Ω . Then there exists $G \in U$ and $F \in V$ such that $G \cap F = \emptyset$ and $\Omega^*_{F^c}$ and $\Omega^*_{G^c}$ are disjoint open set in T containing V and U respectively.

3.55 <u>Lemma</u>. (Ω, T) is compact.

Proof: Let Φ be a family of closed sets $\mathcal{H}_\alpha$ on Ω with the finite intersection property. We will show that Φ has non empty intersection. Let Ψ be an ultrafilter on Ω containing Φ . We know from the remarks preceding lemma 3.54 that $\mathcal{H}_\alpha$ is a finite union of basic closed sets $\Omega_{\mathcal{F}}$, where $\mathcal{F}$ is a filter on X , i.e. $H_\alpha = \Omega_{\mathcal{F}_{\alpha_1}} \cup \ldots \cup \Omega_{\mathcal{F}_{\alpha_n}}$. Since $\mathcal{H}_\alpha \in \Psi$, Ψ contains some $\Omega_{\mathcal{F}_{\alpha_{i(\alpha)}}} = \Omega_{\mathcal{F}_\alpha}$ for each α , by lemma 1.30. Hence these $\Omega_{\mathcal{F}_\alpha}$ have the finite intersection property. If $\bigcap_\alpha \Omega_{\mathcal{F}_\alpha}$ is empty, then no ultrafilter on X contains all the filters $\mathcal{F}_\alpha$. Hence the family of unions of all the $\mathcal{F}_\alpha$ does not have the finite intersection property. Hence for some $F_1 \in \mathcal{F}_{\alpha_1}, \ldots, F_n \in \mathcal{F}_{\alpha_n}$,

$F_1 \cap F_2 \cap \ldots \cap F_n$ must be empty. Then the finite union of the filters $\mathfrak{F}_{\alpha_1}, \ldots, \mathfrak{F}_{\alpha_n}$ does not have the finite intersection property. Contrary to the fact that $\bigcap_{i=1}^{n} \Omega_{\mathfrak{F}_{\alpha_i}}$ is non empty.

Hence the ultrafilter space $(\Omega, \mathfrak{T})$ is compact.

J. COMPACTIFICATIONS USING FILTERS AND ULTRAFILTER SPACES

3.56 Definition. A topological space $\overline{X}$ is a compactification of a topological space X, if $\overline{X}$ is compact and contains a dense subset which is homeomorphic to X.

We shall show, following Samuel's [38] use of ultrafilter spaces, how to construct a compactification for an arbitrary uniformizable space. Samuel required that the given space be Hausdorff, however we shall see that this restriction is not necessary. We shall also see how Wagner [45] has used filter spaces to obtain the well known Alexandroff compactification for locally compact Hausdorff spaces.

Let (X, σ) be a topological space such that the topology σ is compatible with the uniform structure G for X which has a symmetric base $\mathfrak{F}$. Samuel has shown that if we identify those ultrafilters which converge to $x \in X$ and put X in a one-one

correspondence with a subspace of an <u>identification</u> space for X, then X will be homeomorphic to a dense subset of a compact space.

3.57 <u>Definition.</u> Let (X,σ) be a topological space and R an equivalence relation on X. Then the set of all equivalence classes X/R is the <u>identification</u> space of X.

3.58 <u>Definition.</u> A subset A of X is <u>saturated</u> if it is a union of equivalence classes, i.e., if $f^{-1}(f(A)) = A$ where f maps every element of X onto its equivalence class.

We will say that a subset A of X/R is open (closed) if and only if it is the image under f of a saturated open (closed) set of X. Clearly f, from X to X/R, is continuous with this topology. We will refer to this topology as the identification space topology.

Now we shall produce a compact set $\bar{X}$ such that X is homeomorphic to a dense subset of $\bar{X}$. Let Ω be the ultrafilter space of X. We will consider two ultrafilters to be equivalent if and only if they have the same envelope (see definition 2.37). In the ultrafilter space, Ω, let us identify the collection of equivalent ultrafilters and denote this collection by $\bar{X}$. Let the topology on $\bar{X}$ be the identification topology. Then since Ω is compact and the mapping g which assigns each element of Ω to its equivalence class is continuous, $\bar{X}$ is compact.

Next we must find a dense subset $\tilde{X}$ of $\overline{X}$ such that $\tilde{X}$ is homeomorphic to X. Let $\tilde{x}$ designate the collection of all ultra filters on X which converge to x. Then $\tilde{x}$ is an equivalence class and $\tilde{x} \in \overline{X}$. Let $\tilde{X}$ be the collection of all such equivalence classes. Clearly $\tilde{X} \subseteq \overline{X}$. Every open set of Ω, and, in particular, every saturated open open set, contains an Ω_A, which contains a convergent ultrafilter. Hence $\tilde{X}$ is dense in $\overline{X}$.

$\tilde{X}$ is homeomorphic to X, where e is a homeomorphism from X to $\tilde{X}$ which assigns to every $x \in X$ the class of ultrafilters which converges to x. Hence the following:

3.59 <u>Theorem.</u> Every uniformizable space can be embedded into a compact space.

Turning to filter space, we will now outline how Wagner [45] constructed the Alexandroff compactification using filter spaces. He also showed how to obtain the Čech compactification by using filter spaces.

The Alexandroff compactification $\overline{X}$ of a locally compact Hausdorff space X is a compact space such that X is dense in $\overline{X}$ and $\overline{X} - X$ has exactly one point.

3.60 <u>Theorem.</u> If X is a locally compact Hausdorff space then the Alexandroff compactification of X is $\varphi_A = \mathfrak{E} \cup \{\mathfrak{F}\}$ where $\mathfrak{E} = \{ V(x) \mid x \in X \}$ and $\mathfrak{F} = \{F \subset X \mid X - F\}$ is compact.

Proof: Let U be an ultrafilter on X which does not converge on X. Let C be a compact subset of X. If $C \in U$, then the family $U' = \{ F \cap C \mid F \in U \}$ is an ultrafilter on C. Since C is compact U' converges on C. Hence U converges on X. Therefore $C \notin U$ and so $X - C \in U$ and U is a refinment of $\mathcal{F}$.

Also φ_A is regular; Since every basis open set on φ_A is of the form $r(G) = \{\mathcal{H} \in \varphi \mid G \in \mathcal{H}\}$, where G is open in X. X is regular, therefore there exists an open set whose closure, V, is contained in G. It is easy to verify that $r(V) \subseteq r(G)$ and $r(U)$ is closed in φ_A.

Hence by lemma 3.52 φ_A is compact.

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